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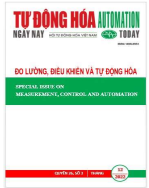

Leaderless- and Leader-Follower Matrix-Weighted Consensus with Uncertainties

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Abstract

This paper considers the matrix-weighted consensus problem with different assumptions on the agent's dynamical model (single-, double integrator with uncertainty or deterministic disturbance) and on the interaction topologies (leaderless and leader-follower graphs). Several decentralized control laws are proposed to make the multi-agent systems asymptotically reach a consensus and reject the effect of the uncertainty and deterministic disturbance. For each proposed consensus law, mathematical analysis is given and reinforced by numerical simulations.

Keywords: *control systems; sliding-mode control; multi-agent systems; graph theory*

Symbols

Symbols	Description
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^d$, $\mathbb{R}^{m\times n}$	the set of d-dimensional vectors and $m \times n$ matrices with real entries
$a,b,c,\alpha,\beta,\gamma,\ldots$	scalars
$\mathbf{a},\mathbf{b},\mathbf{c},\ldots$	vectors
$\mathbf{A},\mathbf{B},\mathbf{C},\ldots$	matrices
$\ker(\mathbf{A})$, $\mathrm{im}(\mathbf{A})$	the kernel and image of matrix $\mathbf{A}$
$\mathbf{I}_n$	the identity matrix of order n
$\mathbf{1}_n$	the vector in $\mathbb{R}^n$ with all entries 1
$\mathbf{0}_{n\times m}$	the $n \times m$ zero matrix
$\mathbf{A}^\top$	the transpose of matrix $\mathbf{A}$
$\mathbf{A} \geq 0$ ($\mathbf{A} > 0$)	matrix $\mathbf{A}$ is positive semidefinite (resp., positive definite)
$\mathrm{vec}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$	$[\mathbf{x}_1^\top,\ldots,\mathbf{x}_n^\top]^\top$
$\mathrm{diag}(\mathbf{x})$	the diagonal matrix with diagonal entries in $\mathbf{x}$
$\mathrm{blkdiag}(\mathbf{A}_1,\ldots,\mathbf{A}_m)$	the block diagonal matrix with diagonal blocks $\mathbf{A}_1,\ldots,\mathbf{A}_m$
$\otimes$	Kronecker product
$\lVert\mathbf{x}\rVert_p$	the p-norm of a vector $\mathbf{x}$ ($p \geq 1$)
$\lVert\mathbf{x}(t)\rVert_{\mathscr{L}_\infty}$	the ∞ norm of the function $\mathbf{x}$, which is equal to $\sup_{t\geq 0}\lVert\mathbf{x}(t)\rVert_\infty$

Abbreviations

MASs	multi-agent systems
MWC	matrix-weighted consensus
SMC	sliding-mode control

Tóm tắt

Nội dung của bài báo này xoay quanh vấn đề đồng thuận trong hệ đa tác tử, trong đó tương tác giữa các tác tử được mô tả bởi một đồ thị trọng số ma trận. Chúng tôi xem xét bài toán với những giả thuyết khác nhau về mô hình của tác tử (khâu tích phân bậc nhất hoặc khâu tích phân kép bị ảnh hưởng bởi nhiễu tiền định và bất định mô hình dạng cộng) và cấu trúc tương tác giữa các tác tử (không có tác tử dẫn đường hoặc tác tử dẫn đường - tác tử theo sau). Một số luật đồng thuận phi tập trung được đề xuất để đưa mọi tác tử về không gian đồng thuận đồng thời loại bỏ ảnh hưởng của bất định mô hình và nhiễu tiền định. Bài báo trình bày các chứng minh toán học và kết quả mô phỏng tương ứng với mỗi luật đồng thuận được đề xuất.

1. Introduction

Recently, consensus algorithms and their applications have received much research attention. Notable applications of the consensus algorithm on the control of multi-agent systems include network synchronization, formation control, distributed optimization, and modeling of social influence networks.
In the consensus algorithm, each agent has a state variable,

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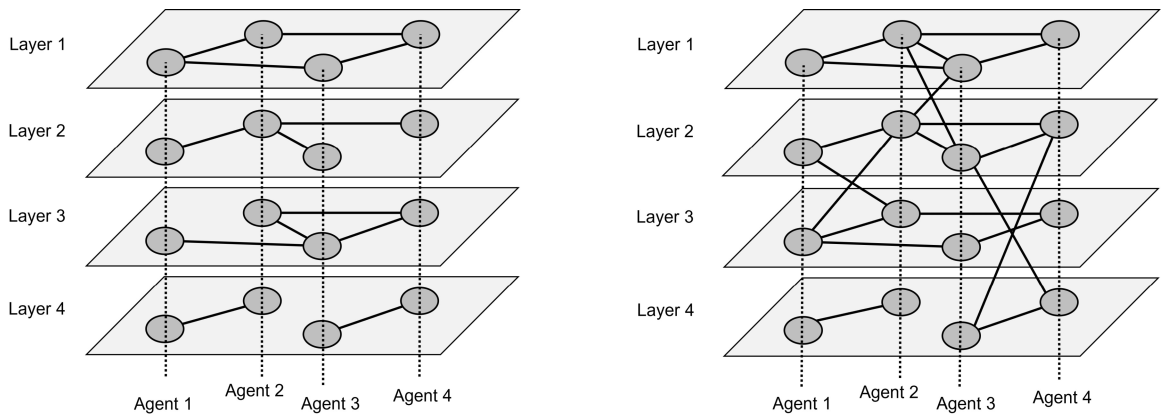


(a) Multiplex network **(b)** Matrix-weighted network

Figure 1: Examples of multiplex- and matrix-weight networks. There are 4 subsystems (or agents) in each network. Each agent has 4 state variables classified into 4 layers. Each state variable is denoted by small circles. A segment connecting two nodes represents an interconnection between the state variables of two agents. A cross-layer interconnection connects two nodes from different layers. In multiplex networks, interconnections between nodes from the same layer. In matrix-weighted networks, cross-layer interconnections between agents are allowed.

which can be updated based on a weighted sum of the relative states with regard to its neighboring agents. The information flow of the network, or in other words, the pattern of interactions between the agents, is usually captured by a graph which is possibly weighted by different scalar weights. It is well-known that the state variables of all agents asymptotically converge to a common value if and only if the graph is connected.

Multiplex networks have been proposed to model multi-agent systems, in which each agent possesses a $d \geq 2$-dimensional state vector divided into d layers. The interactions between agents (subsystems) are separated for each layer and can be characterized by d different scalar-weighted graphs [11, 15]. A typical example of multiplex network is a traffic network between different cities, in which each layers of the network represents a distinguished means of transportation (for examples, air-way, road, waterway, railway, and subway).

From a simple reasoning, it is natural to extend the definition of the multiplex network by allowing cross-layer interactions [4] (see Figure 1 for an illustration of these concepts). It turns out that cross-layer interactions can have several physical interpretations in control of multi-agent systems. For example, the authors in [2] considered a distributed estimation problem, in which a covariance matrix is associated with each edge, and defined the term *matrix-weighted Laplacian*. Analogies between a network with matrix weights and an electrical network were derived in [3]. Matrix-weighted Laplacian was also found in [25], where the authors studied the synchronization of multi-dimensional coupled mechanical and electrical oscillators. In bearing-based formation control and network localization, the rigidity of a framework can be determined from the bearing Laplacian, of which each block entry is an orthogonal projection matrix [32, 33]. An opinion dynamic model was proposed in [1], where the strength of interactions between individuals are captured by state-dependent matrix weights.

Matrix-weighted graphs with positive semidefinite matrix weights and matrix weighted consensus algorithm were firstly studied in [23]. Several algebraic graph conditions for consensus and clustering behaviors of the network under the matrix-weighted consensus algorithm was also proposed. Under the matrix-weighted consensus algorithm, connectedness of the graph does not ensure the agents to reach a consensus. Moreover, the system may asymptotically achieve a configuration in which disagreement between neighboring agents and consensus between agents without direct interactions happen at the same time (see Figure 2). The asymptotic behavior of a system under the matrix-weighted consensus algorithm depends jointly on the distribution and actual values of the matrix weights in the graph.

The authors in [12] provided several graphical conditions for consensus and clustering on a matrix-weighted network. Controllability and observability of matrix-weighted networks were studied in [19, 26]. Methods to synthesize matrix-weighted networks based on some performance indices of the system are considered in [7, 10]. Several discrete-time matrix-weighted consensus algorithms were proposed in [16, 22]. Further extensions, including matrix-weighted consensus with switching network topologies, or bipartite matrix-weighted consensus have been recently examined, for examples, see [8, 14, 20, 28].

Most existing works only consider the leaderless- or the leader-follower matrix-weighted consensus problem with ideal agents' dynamics and with stationary leader agents. In many scenarios, the effect of model's uncertainties or deterministic disturbances from the environment on the agreement dynamics are unavoidable. Specifically, when we restrict the graph to be undirected (or leaderless), the vector of uncertainties and disturbances can be decomposed into two parts: the first part lies along the consensus space and the second part is orthogonal to the consensus space. If each agent can sense their relative states, the first part of the vector made the system drift while the second part can be compensated by an appropriated control law [31]. Decentralized sliding-mode based controllers and observers have been proposed for scalar-weighted consensus under different assumptions on agents' dynamic model, for examples, see [9, 27, 30] and the references therein. In this paper, we firstly consider leaderless (undirected) matrix-weighted networks and propose consensus laws for single- and double-integrator agents with model uncertainties. Under the mild assumption that the kernel of the matrix-weighted Laplacian contains only the consensus space, we show that the system can

$$\boldsymbol{W}_{25} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$\boldsymbol{W}_{12} = \boldsymbol{W}_{24} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\boldsymbol{W}_{13} = \boldsymbol{W}_{34} = \boldsymbol{W}_{35} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

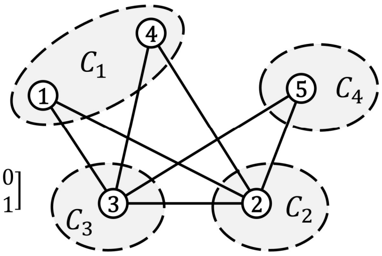


Figure 2: A matrix weighted graph of 5 vertices and 6 edges. Under the matrix-weighted consensus algorithm, the agents converge to four different clusters (denoted by dashed lines) for almost all initial conditions.

asymptotically reach the consensus space under the proposed algorithms. Next, for leader-follower typologies, we show that the leaders' velocity can be considered as a disturbances acting on the system, which drives the system from the consensus space. The control law designed for agents, thus, should be able to reject the disturbance [5, 13]. We propose consensus tracking laws for single- and double-integrator follower agents under the assumption that the leaders move with a common velocity, which is uniformly continuous bounded. It is then shown that the followers asymptotically achieve velocity consensus with the leaders. Position consensus of the system is asymptotically achieved if the leaders are initiated at the same location. In case the leaders are not initially rendezvoused, the followers are asymptotic to different trajectories that jointly depend on the leaders' positions and the matrix weights. There is no guarantee that the agents positions will be contained in the convex hull of the leaders' positions as in the scalar weight case. Thus, this paper extends the result presented in our conference paper [18] by both the graph topology and the agents' dynamic models.

The remainder of this paper is organized as follows. Section 2 gives background on matrix-weighted graphs and formulates the problem. Sections 3 and 4 propose matrix-weighted consensus algorithms for matrix-weighted networks with leaderless and leader-follower topologies and provide corresponding convergence proofs. Simulation results for proposed consensus laws are then provided in Section 4, and finally, Section 5 concludes the paper.

2. Preliminaries

2.1. Matrix-weighted graphs

An undirected matrix weighted graph $\mathscr{G}$ is defined by a triple $= (\mathscr{V}, \mathscr{E}, \mathscr{W})$, where $\mathscr{V} = \{1, \ldots, n\}$ is the vertex set, $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ is the edge set,[1] and $\mathscr{W} = \{\mathbf{W}_{ij}\}_{(i,j)\in\mathscr{E}}$ is the set of symmetric positive definite or positive semidefinite matrix weights corresponding to the edges in $\mathscr{E}$. Clearly, if $d = 1$, we get the usual scalar weighted graph. For undirected graphs (or leaderless topology), it is assumed that $\mathbf{W}_{ij} = \mathbf{W}_{ji}$.

The matrix weighted adjacency matrix of $\mathscr{G}$ is a block matrix $\mathbf{A} = [\mathbf{A}_{ij}] \in \mathbb{R}^{dn\times dn}$, with the block entries

$$\mathbf{A}_{ij} = \begin{cases} \mathbf{W}_{ji}, & (j,i) \in \mathscr{E}, \\ \mathbf{0}_{d\times d}, & (j,i) \notin \mathscr{E}. \end{cases}$$

[1] In this paper, self-loops of the form (i,i) are excluded in the graph.

For each vertex $i \in \mathscr{V}$, the neighbor set of i is defined as $\mathscr{N}_i = \{j \in \mathscr{V} |\ (j,i) \in \mathscr{E}\}$. The degree of the vertex i and the matrix-weighted degree matrix are defined as $\mathbf{D}_i = \sum_{j=1}^{n} \mathbf{A}_{ij}$ and $\mathbf{D} = \text{blkdiag}(\mathbf{D}_1, \ldots, \mathbf{D}_n)$, respectively. Most properties of the matrix-weighted graph are related with the matrix-weighted Laplacian, which is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$. For leaderless topology, $\mathbf{L} \in \mathbb{R}^{dn\times dn}$ is symmetric, positive semidefinite, and $\ker(\mathbf{L}) \supseteq \text{im}(\mathbf{1}_n \otimes \mathbf{I}_d)$.

From the vertex set $\mathscr{V}$, we consider a partition $\{\mathscr{V}_L, \mathscr{V}_F\}$ satisfying $\mathscr{V}_L \neq \emptyset$, $\mathscr{V}_F \neq \emptyset$, $\mathscr{V} = \mathscr{V}_L \cup \mathscr{V}_F$ and $\mathscr{V}_L \cap \mathscr{V}_F = \emptyset$. Without loss of generality, we can label the vertices so that $\mathscr{V}_L = \{1, \ldots, l\}$, $\mathscr{V}_F = \{l+1, \ldots, n\}$, and $f = n - l$, where $1 \leq l \leq n-1$.

According to this partition of $\mathscr{V}$, we can express the matrix-weighted Laplacian in the following form

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{LL} & \mathbf{L}_{LF} \\ \mathbf{L}_{FL} & \mathbf{L}_{FF} \end{bmatrix}, \tag{1}$$

where $\mathbf{L}_{LL} \in \mathbb{R}^{dl\times dl}$, $\mathbf{L}_{LF} = \mathbf{L}_{LF}^{\top} \in \mathbb{R}^{dl\times df}$, and $\mathbf{L}_{FF} \in \mathbb{R}^{df\times df}$. Several useful properties of the matrix-weighted graphs, of which proofs were provided in [18, 23, 24], are summarized in the following lemma.

Lemma 1. *Suppose that rank*$(\mathbf{L}) = dn - d$ *and* $d \geq 1$, *then*

(i) *ker*$(\mathbf{L}) = $ *im*$(\mathbf{1}_n \otimes \mathbf{I}_d)$;

(ii) *Let the eigenvalues of* $\mathbf{L}$ *be given as* $0 = \lambda_1 = \ldots = \lambda_d < \lambda_{d+1} \leq \ldots \leq \lambda_{dn}$, *there exists a positive definite matrix* $\mathbf{P}$ *such that* $\mathbf{P} = \mathbf{P}^{\top} > 0$ *and* $\mathbf{L} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{\top}$;

(iii) $\mathbf{L}_{FF}$ *is symmetric and positive definite;*

(iv) $\mathbf{D}_i$, $i = 1, \ldots, n$, *is symmetric, positive definite;*

(v) *Each block row sum of the matrix* $\mathbf{R} = -\mathbf{L}_{FF}^{-1}\mathbf{L}_{FL}$ *equals to* $\mathbf{I}_d$.

2.2. Problem formulation

Consider a system of $n \geq 2$ agents interacting over a matrix-weighted graph $\mathscr{G}$. To characterize the system, let each agent i be associated with a vertex in $\mathscr{V}$ and a state vector $\mathbf{x}_i \in \mathbb{R}^d$, $d \geq 1$. If agent i has information from agent j, there is an edge $(j,i) \in \mathscr{E}$, with the corresponding matrix weight $\mathbf{W}_{ji} = \mathbf{W}_{ji}^{\top} \geq 0$.

Let $\mathbf{x} = \text{vec}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^{dn}$, the multi-agent system is said to achieve a consensus if and only if $\mathbf{x} \in \ker(\mathbf{L}) = \{\mathbf{x} \in \mathbb{R}^{dn} |\ \mathbf{x}_i = \mathbf{x}_j, \forall i,j \in \mathscr{V}\}$.

For the leaderless interaction topology, we assume that the matrix-weighted Laplacian $\mathbf{L}$ is undirected and satisfies rank$(\mathbf{L}) = dn - d$. The dynamic of n agents are given as follows

(i) single-integrator perturbed by uncertainty and disturbance

$$\dot{\mathbf{x}}_i = \mathbf{u}_i + \mathbf{f}_i(t),\ i = 1, \ldots, n, \tag{2}$$

where $\mathbf{x}_i, \mathbf{u}_i, \mathbf{f}_i(t) \in \mathbb{R}^d$ are respectively the state, the control input, and the vector of uncertainty and disturbance of agent i.

(ii) double-integrator perturbed by uncertainty and disturbance

$$\dot{\mathbf{x}}_i = \mathbf{y}_i,$$
$$\dot{\mathbf{y}}_i = \mathbf{u}_i + \mathbf{f}_i(t),\ i = 1, \ldots, n, \tag{3}$$

where $[\mathbf{x}_i^{\top}, \mathbf{y}_i^{\top}]^{\top} \in \mathbb{R}^{2d}$, $\mathbf{u}_i, \mathbf{f}_i(t) \in \mathbb{R}^d$, are respectively the state, the control input, and the vector of uncertainty and disturbance of agent i.

For both cases, we assume that $\mathbf{f}_i(t)$ satisfies $\|\mathbf{f}_i(t)\|_{\mathscr{L}_\infty} < \gamma$, where $\gamma > 0$. The agents do not know $\mathbf{f}_i(t)$, however, the upper bound γ is available to agent i.

Next, suppose that the interaction graph has a leader-follower topology. There are $l \geq 1$ leaders and $f = n - l$ followers in the system. The interactions between leaders and followers are uni-directional, i.e., only the followers sense the relative states with regard to the leaders. However, the interactions between followers are bi-directional. It follows that the matrix-weighted Laplacian in this case is represented by

$$\mathbf{L} = \begin{bmatrix} \mathbf{0}_{dl\times dl} & \mathbf{0}_{dl\times df} \\ \mathbf{L}_{FL} & \mathbf{L}_{FF} \end{bmatrix}.$$

Let the leaders be modeled by the equation

$$\dot{\mathbf{x}}_i = \mathbf{h}_i(t),\ i = 1,\ldots,l, \tag{4}$$

where $\dot{\mathbf{h}}_i(t)$ are bounded, uniformly continuous functions satisfying $\|\mathbf{h}_i(t)\|_{\mathscr{L}_\infty} < \beta$, $\|\dot{\mathbf{h}}_i(t)\|_{\mathscr{L}_\infty} < \eta$, $\beta, \eta > 0$. The follower agents $i = l+1,\ldots,n$, do not know $\mathbf{h}_i(t)$ but have information on the upper bounds β, η. The followers are modeled by the perturbed single- or double integrators, i.e., the models (2) or (3).[2]

For all types of agent's models studied in this paper, our objective is designing decentralized algorithm for each agent using the local variables ($\mathbf{x}_i$ and/or $\mathbf{y}_i$) and the relative variables $\{\mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j)\}_{j\in\mathscr{N}_i}$ and/or $\{\mathbf{A}_{ij}(\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j)\}_{j\in\mathscr{N}_i}$ so that $\mathbf{x} \to \mathscr{C}$, as $t \to +\infty$.

3. Matrix-weighted consensus with leaderless topology

In this section, it is assumed that the agents' interactions are bi-directional and the corresponding matrix-weighted graph is undirected. We will propose matrix-weighted consensus algorithms for networks of single-integrator and double-integrator agents with disturbances and uncertainties.

3.1. Single-integrator agents

Let the agents be modeled by the single integrator model with uncertainties (2). The algorithm designed for each agent is given by

$$\mathbf{u}_i = -k\sum_{j\in\mathscr{N}_i}\mathbf{A}_{ij}\mathrm{sgn}\left(\sum_{j\in\mathscr{N}_i}\mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j)\right),\ i = 1,\ldots,n, \tag{5}$$

where $k > 0$ is a control gain and the signum function $\mathrm{sgn}(\mathbf{x})$ is defined component-wise for each vector $\mathbf{x} = [x_1,\ldots,x_d]^\top \in \mathbb{R}^d$. That is, $\mathrm{sgn}(\mathbf{x}) = [\mathrm{sgn}(x_1),\ldots,\mathrm{sgn}(x_d)]^\top$ and

$$\mathrm{sgn}(x_i) = \begin{cases} -1, & \text{if } x_i < 0, \\ 0, & \text{if } x_i = 0, \\ 1, & \text{if } x_i > 0. \end{cases},\ i = 1,\ldots,d.$$

Then, the n-agent system is given in matrix form as follows:

$$\dot{\mathbf{x}} = -k\mathbf{L}\mathrm{sgn}(\mathbf{L}\mathbf{x}) + \mathbf{f}(t). \tag{6}$$

Let $\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i = \frac{1}{n}(\mathbf{1}_n^\top \otimes \mathbf{I}_d)\mathbf{x}$ be the average state of n agents at time $t \geq 0$. Then $\dot{\bar{\mathbf{x}}} = \frac{1}{n}(\mathbf{1}_n^\top \otimes \mathbf{I}_d)\dot{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^n \mathbf{f}_i = \bar{\mathbf{f}}$. With the variable transformation $\mathbf{q} = \mathbf{x} - \mathbf{1}_n \otimes \bar{\mathbf{x}}$, it follows that $\mathbf{L}\mathbf{q} = \mathbf{L}\mathbf{x}$. We can express the $\mathbf{q}$-dynamic as follows:

$$\dot{\mathbf{q}} = -k\mathbf{L}\mathrm{sgn}(\mathbf{L}\mathbf{q}) + \mathbf{r}(t), \tag{7}$$

where $\mathbf{r} = \mathbf{f} - \mathbf{1}_n \otimes \bar{\mathbf{f}}$ satisfies $(\mathbf{1}_n^\top \otimes \mathbf{I}_d)\mathbf{r} = \mathbf{0}_d$. It follows that $\mathbf{q} \perp \ker(\mathbf{1}_n \otimes \mathbf{I}_d)$.

Since $\mathrm{sgn}(\cdot)$ is discontinuous, we understand the solution of (7) in Fillipov sense [21], i.e.,

$$\dot{\mathbf{q}} \in -k\mathbf{L}K[\mathrm{sgn}](\mathbf{L}\mathbf{q}) + \mathbf{r}(t),$$

where $K[\mathbf{f}](\mathbf{x})$ denotes the Filippov differential inclusion,

$$\mathrm{sgn}(x_i) = \begin{cases} -1, & \text{if } x_i < 0, \\ [-1,1], & \text{if } x_i = 0, \\ 1, & \text{if } x_i > 0. \end{cases},\ \text{for all } i = 1,\ldots,d.$$

Consider the Lyapunov function $V(t,\mathbf{q}) = V(\mathbf{q}) = \frac{1}{2}\mathbf{q}^\top\mathbf{q}$ which is positive definite, radially unbounded. Because for each $x_i \in \mathbb{R}$, we have $x_i\mathrm{K}[\mathrm{sgn}](x_i) = |x_i|$ [21], we have $\mathbf{x}^\top\mathrm{K}[\mathrm{sgn}](\mathbf{x}) = \sum_{i=1}^d |x_i| = \|\mathbf{x}\|_1$. Thus, it follows that

$$\begin{aligned}\dot{V} &\in^{a.e.} \sum_{\boldsymbol{\eta}\in\partial V}\boldsymbol{\eta}^\top\mathrm{K}[\dot{\mathbf{q}}] = \mathbf{q}^\top(-k\mathbf{L}\mathrm{K}[\mathrm{sgn}](\mathbf{L}\mathbf{q}) + \mathbf{r}(t)) \\ &= -k\|\mathbf{L}\mathbf{q}\|_1 + \mathbf{q}^\top\mathbf{r}(t).\end{aligned} \tag{8}$$

Since $\mathbf{q} \perp \ker(\mathbf{L})$, we have $\|\mathbf{L}\mathbf{q}\|_1 \geq \|\mathbf{L}\mathbf{q}\| \geq \lambda_{d+1}\|\mathbf{q}\| \geq \sqrt{2}\lambda_{d+1}\|\mathbf{q}\|_1$. It follows that

$$\begin{aligned}\dot{V} &= -k\|\mathbf{L}\mathbf{q}\|_1 - \mathbf{q}^\top\mathbf{f} \\ &\leq -\sqrt{2}k\lambda_{d+1}\|\mathbf{q}\|_1 + \|\mathbf{q}\|_1\|\mathbf{f}\|_\infty \\ &\leq -\sqrt{2}k\lambda_{d+1}\|\mathbf{q}\|_1 + \|\mathbf{q}\|_1\|\mathbf{f}\|_{\mathscr{L}_\infty} \\ &\leq -(\sqrt{2}k\lambda_{d+1} - \alpha)\|\mathbf{q}\|_1 \\ &\leq -\kappa V^{\frac{1}{2}},\end{aligned} \tag{9}$$

where $\kappa = \sqrt{2}(\sqrt{2}k\lambda_{d+1} - \alpha) > 0$. To show finite-time convergence of V to 0, we follow the discussion in [17][Thm. 1]. If $V(t) > 0, \forall t \geq 0$, then it follows from (9) that

$$\frac{1}{2}\int_{V(0)}^{V(t)}\frac{dV}{\sqrt{V}} \leq -\frac{\kappa}{2}\int_0^t d\tau, \tag{10}$$

or i.e.,

$$0 \leq \sqrt{V(t)} \leq \sqrt{V(0)} - \frac{\kappa}{2}(t-0). \tag{11}$$

When t is large enough, the right hand side of the inequality (11) becomes negative, which causes a contradiction. This contradiction implies that $\exists T > 0 : V(t) = 0$ and for $t \geq T$. This implies that

$$\mathbf{L}\mathbf{q}(t) = \mathbf{L}\mathbf{x}(t) = \mathbf{0},$$

for $t \geq T$, i.e., the $\mathbf{x}(t) \in \ker(\mathbf{L})$ for $t \geq T$. Thus, the system achieves finite time consensus.

3.2. Double-integrator agents

Let the agents be modeled by the double integrator with uncertainties (3). Define the error variables

$$\mathbf{e}_i = \mathbf{y}_i + \sum_{j\in\mathscr{N}_i}\mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j),$$

[2] We can interpret $\mathbf{x}_i$ and $\mathbf{y}_i$ as the position and velocity of agent i in the d-dimensional space, respectively.

and the sliding manifolds

$$\mathbf{s}_i = \mathbf{e}_i + \lambda \int_0^t \mathrm{sig}^{\alpha}(\mathbf{e}_i(\tau))d\tau \equiv \mathbf{0}_d,\ i = 1,\ldots,n, \tag{12}$$

where $\lambda > 0$ is a positive scalar and $\alpha \in (0,1)$. Defining the stacked variables $\mathbf{s} = \mathrm{vec}(\mathbf{s}_1,\ldots,\mathbf{s}_n)$, $\mathbf{e} = \mathrm{vec}(\mathbf{e}_1,\ldots,\mathbf{e}_n)$, we have

$$\mathbf{e} = \mathbf{y} + \mathbf{L}\mathbf{x}, \tag{13}$$

$$\mathbf{s} = \mathbf{e} + \lambda \int_0^t \mathrm{sig}^{\alpha}(\mathbf{e}(\tau))d\tau \equiv \mathbf{0}_{dn}. \tag{14}$$

It follows that we need to design $\mathbf{u}$ such that

$$\dot{\mathbf{s}} = \mathbf{u} + \mathbf{L}\mathbf{y} + \mathbf{f} + \lambda \mathrm{sig}^{\alpha}(\mathbf{y} + \mathbf{L}\mathbf{x}) \equiv \mathbf{0}_{dn}. \tag{15}$$

Consider the Lyapunov function $V_1 = \frac{1}{2}\mathbf{s}^\top\mathbf{s}$, we have

$$\dot{V}_1 = \mathbf{s}^\top\left(\mathbf{u} + \mathbf{L}\mathbf{y} + \mathbf{f} + \lambda \mathrm{sig}^{\alpha}(\mathbf{y} + \mathbf{L}\mathbf{x})\right).$$

Let the control law be designed as

$$\mathbf{u} = -k\mathrm{sgn}(\mathbf{s}) - \mathbf{L}\mathbf{y} - \lambda \mathrm{sig}^{\alpha}(\mathbf{y} + \mathbf{L}\mathbf{x}), \tag{16}$$

where $k > \gamma$, it follows that

$$\begin{aligned}\dot{V}_1 \in \dot{\tilde{V}}^{a.e.} &= -k\|\mathbf{s}\|_1 - \mathbf{s}^\top\mathbf{f} \\ &\leq -(k - \|\mathbf{f}\|_\infty)\|\mathbf{s}\|_1 \\ &\leq -\sqrt{2}(k-\gamma)V^{\frac{1}{2}},\end{aligned} \tag{17}$$

and this implies the existence of $T_1 \geq 0$ such that $\mathbf{s}(t) \equiv \mathbf{0}_{dn}, \forall t \geq 0$. This implies $\dot{\mathbf{e}} = -\lambda \mathrm{sig}^{\alpha}(\mathbf{e})$ for $t \geq T_1$.
Next, consider the function $V_2 = \frac{1}{2}\mathbf{e}^\top\mathbf{e}$, for $t \geq T_1$, we have

$$\begin{aligned}\dot{V}_2 &= -\lambda \mathbf{e}^\top \mathrm{sig}^{\alpha}(\mathbf{e}) \\ &= -\lambda \sum_{i=1}^{n}\sum_{k=1}^{d} |\mathbf{e}_{ik}|^{\alpha+1} \\ &\leq -\lambda \sum_{i=1}^{n}\sum_{k=1}^{d} |\mathbf{e}_{ik}^2|^{\frac{\alpha+1}{2}}.\end{aligned} \tag{18}$$

By applying Lemma 2 for $\frac{1}{2} < \frac{\alpha+1}{2} < 1$, it follows that $\sum_{i=1}^{n}\sum_{k=1}^{d}|\mathbf{e}_{ik}^2|^{\frac{\alpha+1}{2}} \geq \left(\sum_{i=1}^{n}\sum_{k=1}^{d}|\mathbf{e}_{ik}^2|\right)^{\frac{\alpha+1}{2}} = 2^{\frac{\alpha+1}{2}}V_2^{\frac{\alpha+1}{2}}$. Thus, it follows from (18) that

$$\dot{V}_2(t) \leq -\kappa_2 V_2^{\frac{\alpha+1}{2}}(t),\ \forall t \geq T_1, \tag{19}$$

where $\kappa_2 = \lambda 2^{\frac{\alpha+1}{2}}$. This implies the existence of $T_2 \geq T_1$ such that $\mathbf{e}(t) = \mathbf{0}_{dn}$ for all $t \geq T_2$.
Finally, for $t \geq T_2$, $\mathbf{y} = -\mathbf{L}\mathbf{x}$ and thus, we arrive at the usual matrix-weighted consensus equation $\dot{\mathbf{x}}(t) = -\mathbf{L}\mathbf{x}(t), \forall t \geq T_2$. Thus, $\mathbf{x}(t) \to \ker(\mathbf{L})$ as $t \to +\infty$.

4. Matrix-weighted consensus with leader-follower topology

In this section, we study the matrix-weighted consensus tracking problem when the interaction graph is given by a leader-follower topology. That is, there are several leaders whose follow some deterministic reference trajectories. Two models of the followers will be considered in this section including the single-integrator and the double-integrator models perturbed with uncertainties. In order to achieve a consensus, the followers needs to track the leaders' position.

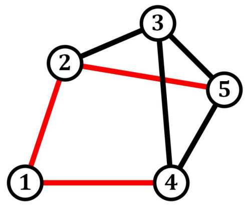


Figure 3: A matrix weighted graph of 5 vertices and 7 edges. Red (respectively, black) denotes that the edge has a corresponding positive definite (respectively, positive semidefinite) matrix weight.

4.1. Single-integrator follower agents

Let $\mathbf{x}^l = \mathrm{vec}(\mathbf{x}_1,\ldots,\mathbf{x}_l)$, $\mathbf{x}^f = \mathrm{vec}(\mathbf{x}_{l+1},\ldots,\mathbf{x}_n)$, $\mathbf{v}^l = \mathrm{vec}(\mathbf{v}_1,\ldots,\mathbf{v}_l) = \mathbf{1}_l \otimes \mathbf{h}$, and $\mathbf{v}^f = \mathrm{vec}(\mathbf{v}_{l+1},\ldots,\mathbf{v}_n)$. In case the followers are modeled by single integrators, we can write the multi-agent systems in the following form

$$\begin{bmatrix}\dot{\mathbf{x}}^l \\ \dot{\mathbf{x}}^f\end{bmatrix} = \bar{\mathbf{Z}}\begin{bmatrix}\mathbf{0}_{dl} \\ \mathbf{u}^f + \mathbf{f} - \mathbf{1}_f \otimes \mathbf{h}(t)\end{bmatrix} + \mathbf{1}_n \otimes \mathbf{h}(t),$$

where $\mathbf{Z} = \begin{bmatrix}\mathbf{0}_{l\times l} & \mathbf{0}_{l\times f} \\ \mathbf{0}_{f\times l} & \mathbf{0}_{f\times f}\end{bmatrix}$ and $\bar{\mathbf{Z}} = \mathbf{Z} \otimes \mathbf{I}_d$.
Let $\mathbf{x}^L = \mathrm{vec}(\mathbf{x}_1,\ldots,\mathbf{x}_l)$ and $\mathbf{x}^F = \mathrm{vec}(\mathbf{x}_1,\ldots,\mathbf{x}_l)$. Since only the followers are responsible for consensus tracking, we define the position error vectors

$$\begin{aligned}\mathbf{e}_i &= \sum_{j\in\mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j) \\ &= \sum_{j=l+1}^{n} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j) + \sum_{j=1}^{l} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j),\end{aligned} \tag{20}$$

where $i = l+1,\ldots,n$. Then, by defining the vector $\mathbf{e}^F = \mathrm{vec}(\mathbf{e}_{l+1},\ldots,\mathbf{e}_n)$, we have

$$\mathbf{e}^F = \mathbf{L}_{FL}\mathbf{x}^L + \mathbf{L}_{FF}\mathbf{x}^F.$$

If the leaders' state vector satisfies $\mathbf{x}_1(0) = \ldots = \mathbf{x}_l(0)$, it follows that $\mathbf{x}^L = \mathbf{1}_l \otimes \mathbf{x}_1$ and

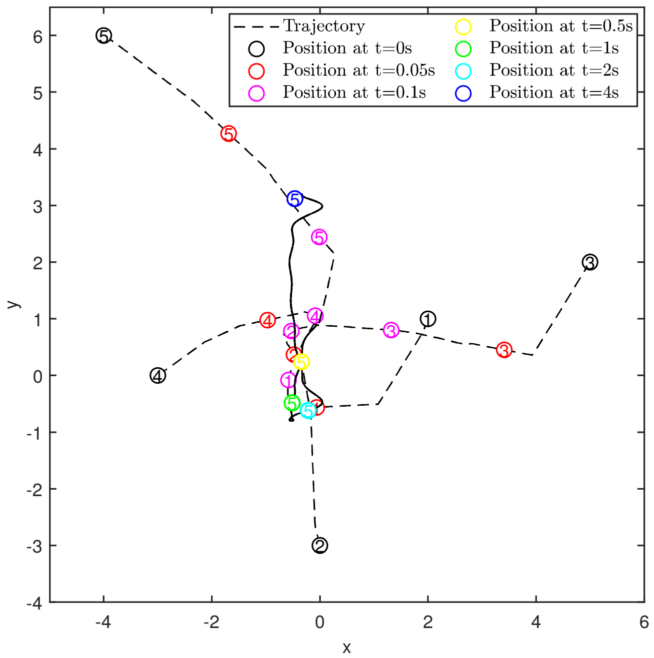
$$\begin{aligned}\mathbf{e}^F &= \mathbf{L}_{FL}(\mathbf{1}_l \otimes \mathbf{x}_1) + \mathbf{L}_{FF}\mathbf{x}^F \\ &= \mathbf{L}_{FL}(\mathbf{1}_l \otimes \mathbf{I}_d)\mathbf{x}_1 + (\mathbf{L}' - \mathrm{blkdiag}\big(\mathbf{L}_{FL}(\mathbf{1}_f \otimes \mathbf{I}_d)\big))\mathbf{x}^F,\end{aligned} \tag{21}$$

where $\mathbf{L}'$ is the matrix-weighted Laplacian associated with the induced sub-graph with vertices in $\mathcal{V}_f$. Due to property of a matrix-weighted Laplacian, $\mathbf{L}'\mathbf{x}^F = \mathbf{L}'(\mathbf{x}^F - \mathbf{1}_f \otimes \mathbf{x}_1)$. Combining with $\mathbf{L}_{FL}(\mathbf{1}_l \otimes \mathbf{I}_d)\mathbf{x}_1 = \mathrm{blkdiag}\big(\mathbf{L}_{FL}(\mathbf{1}_l \otimes \mathbf{I}_d)\big)(\mathbf{1}_f \otimes \mathbf{x}_1)$, it follows that $\mathbf{e}^F = \mathbf{L}_{FF}(\mathbf{x}^F - \mathbf{1}_f \otimes \mathbf{x}_1)$ and $\dot{\mathbf{e}}^F = \mathbf{L}_{FF}(\mathbf{u}^F + \mathbf{f} - \mathbf{1}_f \otimes \mathbf{h})$.
Consider the Lyapunov function $V = \frac{1}{2}(\mathbf{e}^F)^\top \mathbf{L}_{FF}^{-1}\mathbf{e}^F$ which is positive definite and radially unbounded. Then, we have $\frac{1}{2}\lambda_{\max}^{-1}(\mathbf{L}_{FF})\|\mathbf{e}^F\|_2^2 \leq V \leq \frac{1}{2}\lambda_{\min}^{-1}(\mathbf{L}_{FF})\|\mathbf{e}^F\|_2^2$. The time derivative of V is given as follows

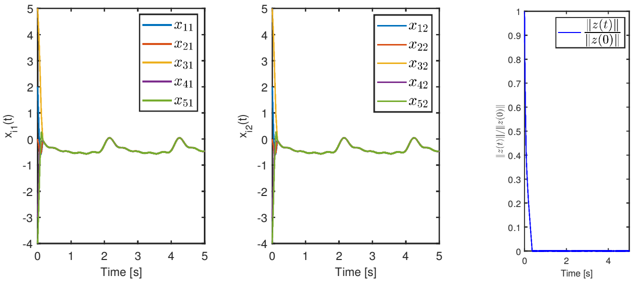
$$\dot{V} = (\mathbf{e}^F)^\top(\mathbf{u}^F + \mathbf{f} - \mathbf{1}_f \otimes \mathbf{h}). \tag{22}$$

The consensus tracking law for the system is then designed as

$$\mathbf{u}^F = -k_1\mathbf{e}^F - k_2\mathrm{sgn}(\mathbf{e}^F), \tag{23}$$



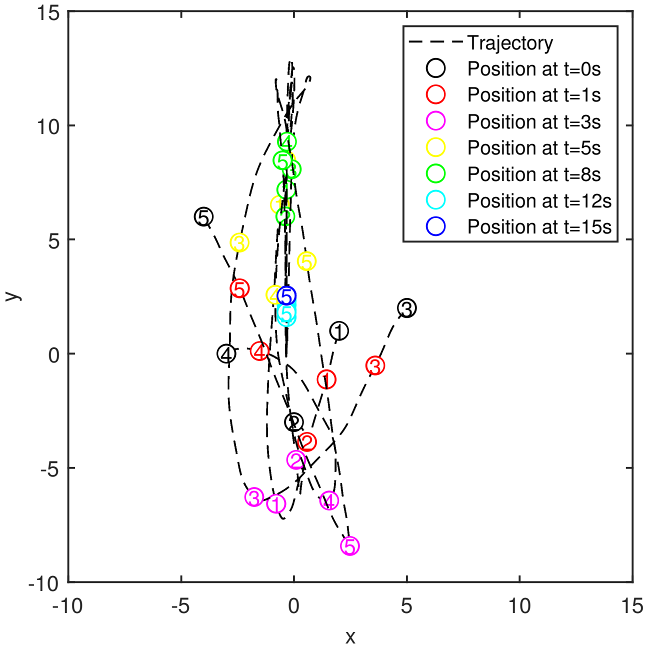
(a) Trajectories of 5 agents



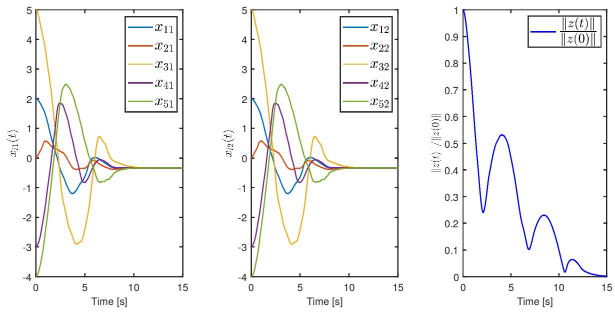
(b) States variables in x- and y- axes

(c) The normalized disagreement vector

Figure 4: The leaderless matrix-weighted consensus system under the control law (6).



(a) Trajectories of 5 agents



(b) Followers' velocities in x-and $y-$axis

(c) The normalized disagreement vector

Figure 5: Simulation results of the 5-agent system under the control law (16).

where $k_1 > 0$ and $k_2 \geq \|\mathbf{h}(t)\|_{\mathscr{L}_\infty} + \|\mathbf{f}(t)\|_{\mathscr{L}_\infty} = \gamma + \beta$. Under the consensus tracking law (23), we have

$$\begin{aligned}
\dot{V} \in \dot{\tilde{V}} &= (\mathbf{e}^F)^\top(-k_1\mathbf{e}^F - k_2\mathrm{sgn}(\mathbf{e}^F) + \mathbf{f} - \mathbf{1}_f \otimes \mathbf{h}) \\
&\leq -k_1\|\mathbf{e}^F\|_2^2 - k_2\|\mathbf{e}^F\|_1 + \|\mathbf{e}^F\|_1(\|\mathbf{f}\|_{\mathscr{L}_\infty} + \|\mathbf{1}_f \otimes \mathbf{h}\|_{\mathscr{L}_\infty}) \\
&\leq -k_1\|\mathbf{e}^F\|_2^2 - (k_2 - \gamma - \beta)\|\mathbf{e}^F\|_1 \\
&\leq -(k_2 - \gamma - \beta)\sqrt{2\lambda_{\min}(\mathbf{L}_{FF})}V^{\frac{1}{2}}.
\end{aligned}$$

This implies the existence of $T > 0$ such that $V(t) = 0$ as $t \geq T$. Therefore, $\mathbf{e}^F = \mathbf{L}_{FL}\mathbf{x}^L + \mathbf{L}_{FF}\mathbf{x}^F = \mathbf{0}$, which implies that $\mathbf{x}^F = -\mathbf{L}_{FF}^{-1}\mathbf{L}_{FL}\mathbf{x}^L = -\mathbf{R}(\mathbf{1}_l \otimes \mathbf{x}_1) = \mathbf{1}_f \otimes \mathbf{x}_1$. Thus, all agents will track the leader's trajectory in finite time.
Note that the consensus tracking law written for each agent is given by

$$\mathbf{u}_i = -k_1 \sum_{j \in \mathscr{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j) - k_2 \mathrm{sgn}\left(\sum_{j \in \mathscr{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j)\right), \quad (24)$$

where $i = l+1, \ldots, n$, and it is distributed in the sense that it uses only relative variables $\{\mathbf{x}_i - \mathbf{x}_j\}_{j \in \mathscr{N}_i}$.

4.2. Double-integrator follower agents

Let the followers be modeled by the perturbed double-integrator models as in (3). We will design a consensus tracking law based on the sliding-mode control approach [18]. Define the error variables as follows

$$\begin{aligned}
\mathbf{e}_{xi} &= \sum_{j \in \mathscr{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j), \\
\mathbf{e}_{yi} &= \sum_{j \in \mathscr{N}_i} \mathbf{A}_{ij}(\mathbf{y}_i - \mathbf{y}_j), \; i = l+1, \ldots, n,
\end{aligned}$$

Then, we have $\dot{\mathbf{e}}_{xi} = \mathbf{e}_{yi}$. Define the sliding variable $\mathbf{s}_i = \lambda \mathbf{e}_{xi} + \mathbf{e}_{yi}$, then $\dot{\mathbf{s}}_i = \lambda \mathbf{e}_{yi} + \dot{\mathbf{e}}_{yi} = \lambda \sum_{j \in \mathscr{N}_i} \mathbf{A}_{ij}(\mathbf{y}_i - \mathbf{y}_j) + \sum_{j \in \mathscr{N}_i} \mathbf{A}_{ij}(\dot{\mathbf{y}}_i - \dot{\mathbf{y}}_j)$. In vector form, we can write

$$\begin{aligned}
\mathbf{e}_x^F &= \mathbf{L}_{FF}(\mathbf{x}^F - \mathbf{1}_f \otimes \mathbf{x}_1), \\
\mathbf{e}_y^F &= \mathbf{L}_{FF}(\mathbf{v}^F - \mathbf{1}_f \otimes \mathbf{h}), \\
\mathbf{s} &= \mathbf{L}_{FF}(\lambda(\mathbf{x}^F - \mathbf{1}_f \otimes \mathbf{x}_1) + \mathbf{v}^F - \mathbf{1}_f \otimes \mathbf{h}), \\
\dot{\mathbf{s}} &= \mathbf{L}_{FF}(\lambda \mathbf{v}^F - \lambda(\mathbf{1}_f \otimes \mathbf{h}) + \mathbf{u}^F + \mathbf{f} - \mathbf{1}_f \otimes \dot{\mathbf{h}}).
\end{aligned}$$

The equivalence control part is given from equation $\dot{\mathbf{s}}|_{\mathbf{h}=\mathbf{0}_{df}, \dot{\mathbf{h}}=\mathbf{0}_{df}} \equiv \mathbf{0}_{df}$ as $\mathbf{u}_{eq}^F = -\lambda \mathbf{v}^F$.

For the normal control part, consider the function $V = \frac{1}{2}\mathbf{s}^\top \mathbf{L}_{FF}^{-1}\mathbf{s}$, we have

$$\dot{V} = \mathbf{s}^\top(\mathbf{u}_N^F - \lambda(\mathbf{1}_f \otimes \mathbf{v}_1) + \mathbf{f} - \mathbf{1}_f \otimes \dot{\mathbf{h}}).$$

Thus, by choosing $\mathbf{u}_N^F = -k_1\mathbf{s} - k_2\mathrm{sgn}(\mathbf{s})$, where $k_1 > 0$, $k_2 > \lambda\beta + \eta$, we have

$$\begin{aligned}\dot{V} &\leq -k_1\|\mathbf{s}\|_2^2 - (k_2 - \lambda\beta - \eta)\|\mathbf{s}\|_1 \\ &\leq -(k_2 - \lambda\beta - \eta)\sqrt{2\lambda_{\min}(\mathbf{L}_{FF})}V^{\frac{1}{2}}. \end{aligned} \tag{25}$$

It follows from equation (25) that $V \to 0$ in finite time. It follows that there exists $T > 0$ such that $\mathbf{s}(t) = \mathbf{0}_{df}$ for $t \geq T$. For $t \geq T$, we have $\lambda\dot{\mathbf{e}}_x + \mathbf{e}_x = \mathbf{0}_d$, which implies $\mathbf{e}_x = \mathbf{L}_{FF}(\mathbf{x}^F - \mathbf{1}_f \otimes \mathbf{h}) \to \mathbf{0}_{df}$. Since $\mathbf{L}_{FF}$ is invertible, it follows that $\mathbf{x}^F \to \mathbf{1}_f \otimes \mathbf{x}_1$ as $t \to +\infty$, or all agents eventually track the leaders' positions.
It is not hard to check that the consensus tracking law of each agent, given as

$$\mathbf{u}_i = -\lambda\mathbf{v}_i - k_1\mathbf{s}_i - k_2\mathrm{sgn}(\mathbf{s}_i), \quad i = l+1, \ldots, n, \tag{26}$$

uses only local and relative variables, and thus, it is decentralized.

5. Simulation results

This section provides simulation results for the proposed matrix-weighted consensus laws in the previous sections.

5.1. Leaderless matrix-weighted consensus

Consider the five-agent system which has the interaction graph $\mathcal{G}$ as depicted in Figure 3. In this graph, matrix weights associated with 7 edges are chosen as follows

$$\mathbf{A}_{12} = 3\mathbf{A}_{25} = 6\mathbf{I}_2,$$
$$\mathbf{A}_{14} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{A}_{23} = 3\mathbf{A}_{34} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix},$$
$$\mathbf{A}_{35} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \mathbf{A}_{45} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

It is not hard to check that the matrix weighted Laplacian of G satisfies $\mathrm{rank}(L) = 2n - 2 = 8$.

5.1.1. Single-integrator agents

We simulate the 5-agent system under the matrix-weighted consensus (6) with $k = 5$. The disturbances are selected as follows

$$\mathbf{f}_i(t) = \begin{bmatrix} 3\cos(3it + \frac{\pi}{3}) \\ -2\cos(t) \end{bmatrix}, \ i = 1, \ldots, 5, \ \forall t \geq 0. \tag{27}$$

Simulation results are shown in Figure 4. Figure 4 (a) shows the trajectories of the agents and denote their positions at several time instances (when we interpret the state vector $\mathbf{x}_i \in \mathbb{R}^2$ of each agent as its absolute position in the x- and $y-$ axes of a 2D plane). Figure 4 (b) shows the states in the x- and y-coordinates and Figure 4 (c) plots the normalized disagreement vector $\frac{\|\mathbf{z}(t)\|}{\|\mathbf{z}(0)\|}$, where $\mathbf{z}(t) = \mathbf{L}\mathbf{x}(t)$, $t \geq 0$. It can be seen that the system achieves a consensus in less than 0.5 second, which is consistent with the analysis.

5.1.2. Double-integrator agents

Next, the same system is considered under the consensus law (16) with $k = 7$ and $\lambda = 1$. The initial velocities of five agents are $\mathbf{v}_i = \mathbf{0}_2$, $i = 1, \ldots, 5$.
The simulation results are provided in Figure 5, with agents' positions being marked at different time instants $t = 0, 1, 3, 5, 8, 12$, and 15 seconds.
All agents asymptotically achieve consensus in both position and velocity after about 15 seconds. After achieving a consensus, their motions are driven by the component of the vector $\mathbf{f} = \mathrm{vec}(\mathbf{f}_1, \ldots, \mathbf{f}_n)$ which lies in the space $\ker(\mathbf{L})$ (and thus perpendicular to $\mathrm{im}(\mathbf{L})$).

5.2. Leader-follower matrix-weighted consensus

5.2.1. Single-integrator followers

We consider the same 5 agent system as in the previous simulations, but agent 1 is now chosen as the leader, moving with a reference velocity given by

$$\mathbf{h}(t) = \begin{bmatrix} 3\cos(3t + \frac{\pi}{3}) \\ -2\cos(t) \end{bmatrix}, \ \forall t \geq 0. \tag{28}$$

The trajectory of the leader is thus a Lissajous curve. The followers are moving under the consensus tracking law (23) with $k_1 = 1$ and $k_2 = 12$.
Simulation results are shown in the Figure 6. The system achieves position consensus after about 0.25 seconds.

5.2.2. Single-integrators with different initial leaders' positions

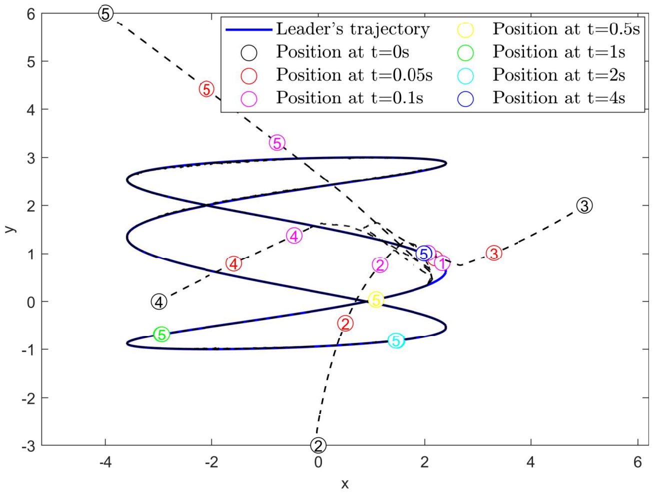
Next, we consider an 8-agent system whose the interaction graph is depicted in Figure 7. The matrix weights are chosen as follows

$$\mathbf{A}_{12} = 2\mathbf{A}_{25} = 6\mathbf{A}_{34} = 6\mathbf{A}_{45} = 6\mathbf{I}_2,$$
$$\mathbf{A}_{14} = \mathbf{A}_{35} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{A}_{23} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix},$$
$$3\mathbf{A}_{57} = \mathbf{A}_{36} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{A}_{56} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{A}_{37} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ \mathbf{A}_{48} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}.$$

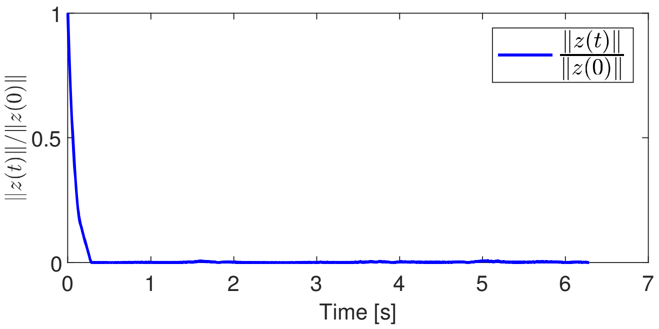
Agents 1, 2, 3 are chosen as leaders and they are initially located at three vertices of a right triangle. The followers adopt the consensus law (23). It is shown in Figure 8 that agents 4, 5 and 8 asymptotically reaches inside the convex hull formed by 3 leaders while agents 6, 7 stay outside. Note that if the graph is scalar-weighted and connected, under the consensus tracking algorithm (23), all followers should asymptotically reach inside the convex hull of leaders' positions, see e.g., the reference [6] on the containment control problem with scalar weighted graph. Thus, this simulation displays a different property of the matrix-weighted consensus in comparison with the scalar-weighted consensus algorithm.

5.2.3. Double-integrator followers

Finally, we return to the 5-agent system with the matrix weighted graph as depicted in Figure 3. The leader (agent



(a) Trajectories of 5 agents



(b) Normalized disagreement vector vs time

Figure 6: The 5-agent leader-follower consensus system under the control law (23).

1) moves with velocity as given in equation (28). Follower agents $2,\ldots,5$ move under the consensus tracking law (26). Simulation results are depicted in Figure 9. The positions of 5 agents at several time instance are depicted in Fig. 9 (a). It can be seen that 4 follower agents asymptotically track and follow the leader's trajectory. The consensus of agents' velocities along two coordinates are shown in Fig. 9 (b). The system reaches the sliding surface after about 1.22 second. Fig. 9 (c) shows the changes of the normalized disagreement vector versus time.

6. Conclusions

This paper studied the matrix-weighted consensus problem with uncertainties for both leaderless and leader-follower topologies. Under different assumptions on the agents' models, several matrix-weighted consensus algorithms were designed with the ability to reject uncertainties or disturbances acting on the system, thus, provide robustness to the matrix-weighted consensus algorithm.

A. Useful lemma

Lemma 2. *[29] If $\xi_1,\ldots,\xi_d \geq 0$ and $0 \leq p \leq 1$, then*

$$\left(\sum_{i=1}^{d}\xi_i\right)^p \leq \sum_{i=1}^{d}\xi_i^p. \tag{29}$$

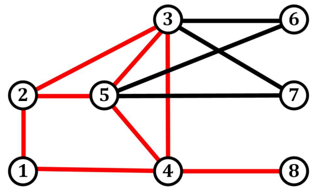


Figure 7: A matrix-weighted graph of 8 vertices ans 12 edges.

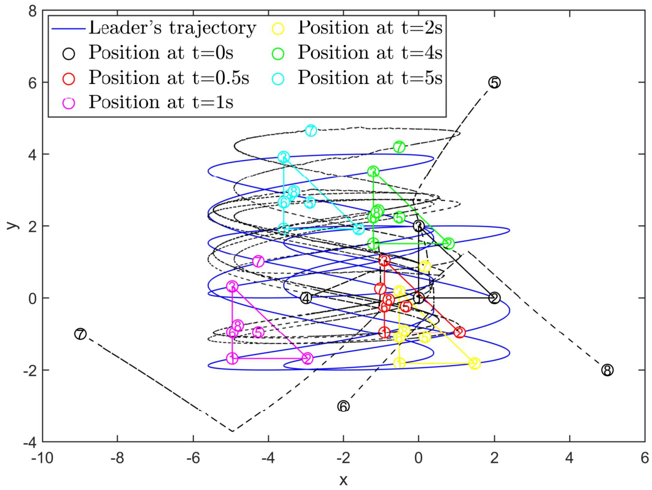


Figure 8: The leader-follower consensus system under the control law (23) when 3 leaders are not initially positioned at the same position. Followers 4, 5, 8 asymptotically move inside the convex hull of leaders' positions (the moving triangle) but agents 6 and 7 do not.

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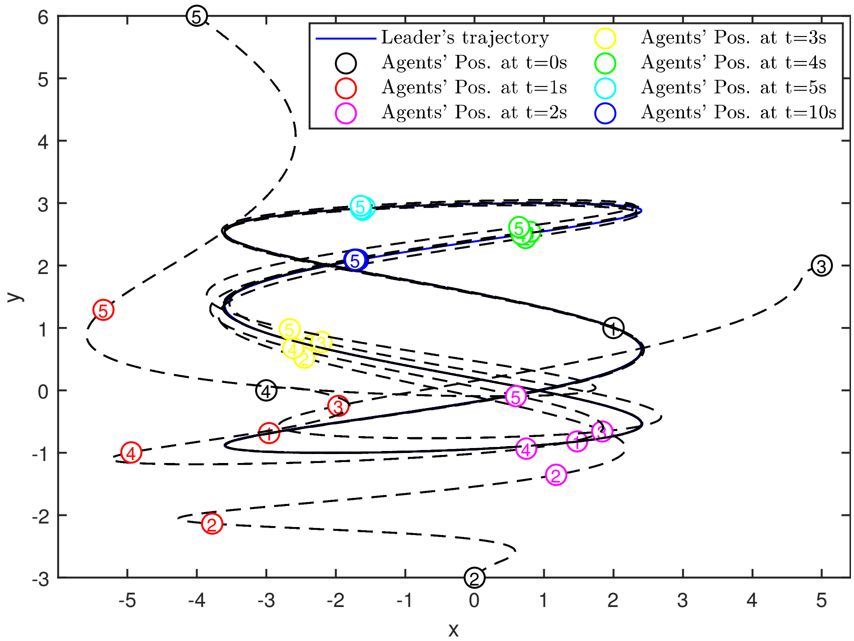
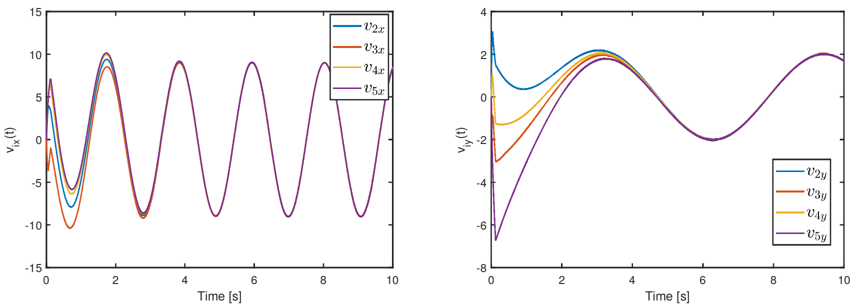
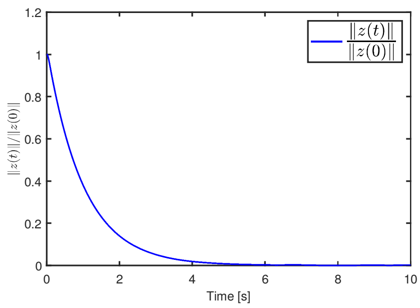


(a) Trajectories of 5 agents



(b) Followers' velocities along two axes of the Oxy plane .



(c) The normalized disagreement vector

Figure 9: The leader-follower consensus system under the control law (26).

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