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Simplifying Nonlinear Control Design with Reduced-Complexity TS Fuzzy System

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Abstract

This work introduces a new method to decrease the complexity of Takagi-Sugeno (TS) representations with sector nonlinearity. Takagi-Sugeno (TS) fuzzy control is a structured method deal with managing non-linear systems, first introduced in 1985. It depends on the utilization of fuzzy system, which represent a collection of If-Then fuzzy regulations with regional linear descriptions in the consequent parts. The proposed method uses linear interconnections among submodels to simplify the TS fuzzy model, resulting in fewer rules and maintaining equivalence to the original model. The simplified model is demonstrated as a combination of $(p+1)$ matrices that are affine, with stability analysis and controller design performed using linear matrix inequalities. The suggested technique has been shown on inverted pendulum model, showing the benefits of reduced complexity for of nonlinear control systems. The reduced-complexity model may result in conservative stability conditions compared to the classical TS fuzzy approach, but offers a significant reduction in numerical complexity and increased computational efficiency for complex systems.

Keywords: *Takagi-Sugeno model, Fuzzy control, Fuzzy rules reduction, Inverted pendulum, Stabilization control.*

1. Introduction

Nonlinear fuzzy control is a methodology for controlling nonlinear systems that involves the use of fuzzy logic concepts to model and control complex, nonlinear behaviors. Nonlinear systems can be difficult to model and control using traditional linear control methods, making nonlinear fuzzy control a valuable tool for addressing these challenges. One of the most commonly used approaches in nonlinear fuzzy control is Takagi-Sugeno control methodology [1] , [2].

A collection of If-Then fuzzy regulations portrays TS fuzzy models, where the consequent parts of the rules are local linear representations. Combining multiple regional linear submodels with membership functions results in a state-space configuration. These models are powerful tools for developing function approximation, system identification, and control design, especially for nonlinear systems that are difficult to model using traditional linear control methods [3] , [4].

TS fuzzy control has a wide range of applications such as robots [5], [6], [7], [8]; the permanent magnet synchronous motor [9], tower crane [10], spacecraft system [11], a hydraulic turbine [12]... A crucial benefit of TS fuzzy control is its methodical strategy for managing non-linear systems. The utilization of this technique relies on mathematical models to represent the system, making it possible to perform stability analysis and design controllers that are well-suited for the specific system. In addition, the use of fuzzy logic concepts provides a means of representing complex, nonlinear behaviors in a way that is more intuitive and easier to understand than traditional linear control methods.

Another advantage of TS fuzzy control is that it offers a robust resolution for creating function approximation and identifying systems. This is because the fuzzy models used in TS fuzzy control can be used to approximate the behavior of the nonlinear system, even when the underlying dynamics are not fully understood. This makes it possible to develop models that are accurate and robust even when dealing with highly nonlinear systems. But the performance of TS fuzzy control can be affected by the choice of membership functions, which can be difficult to tune in real-world applications. Despite its limitations, the use of TS fuzzy control is growing in popularity due to its versatility and ability to handle a wide range of nonlinear systems.

Reducing the number of fuzzy rules in Takagi-Sugeno (TS) control is necessary due to several factors. One of the main factors is to simplify control system design and implementation, especially for real-time implementation in systems that are highly non-linear and contain a vast number of premise variables. This is because having a large number of fuzzy rules can make the design process complex and difficult to implement in real-time. Another reason is that having a large number of fuzzy rules can increase the computational burden on the system, making the control process slow and less efficient. Furthermore, having a large number of fuzzy rules can also lead to overfitting and decreased accuracy in the control process, making it less effective.

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By decreasing the quantity of fuzzy regulations, the complexity of the control system can be reduced, making it more efficient and effective. This can result in improved control performance and stability, making it easier to implement in real-world systems. Additionally, diminishing the quantity of fuzzy regulations can make it easier to analyze the stability and durability of system control, allowing for more accurate and effective control design. Finally, diminishing the quantity of fuzzy regulations can make it easier to extend the control system to new applications, making it more flexible and adaptable to changing requirements.

The primary contribution of this paper is the proposal of a novel method to format the TS fuzzy model while reducing the number of fuzzy rules. By expressing the model in a new format, the number of rules is minimized, which can lead to a more straightforward implementation of the controller. In order to showcase the efficacy of the suggested approach, the inverted pendulum model is utilized as an example, and the simulation results show good performance in stabilizing the system. The simulation results suggest that the new TS fuzzy model format can effectively model complex nonlinear systems and can provide a more straightforward approach to designing a controller.

The next section of this paper focuses on the design and analysis of the TS fuzzy model. The section discusses the process of reducing the model's complexity by applying fuzzy logic, which involves the formulation of a reduced model with a fewer number of rules. Then, the paper presents the modeling and design of a control system using the TS fuzzy model, with a focus on the application in the inverted pendulum system. The section concludes with a detailed presentation of the simulation results, which showcase the efficacy of the suggested approach in stabilizing the inverted pendulum. Finally, the paper provides a conclusion that summarizes the primary results of the study and highlights the significance of the proposed approach in controlling nonlinear systems.

2. Design and Analysis of Takagi-Sugeno Fuzzy Models

The TS system equation is a mathematical representation of a nonlinear system using a fuzzy modeling approach. It is based on a collection of fuzzy rules in the form of "If-Then" statements, where the antecedent parts are the premise variables and the consequent parts are local linear representations of the system. The TS system equation is obtained by combining the local linear representations using a convex sum with weights represented by membership functions. The membership functions determine the contribution of each local linear representation to the overall fuzzy model.

The TS system equation presents as:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{\omega} \eta_i(\mathbf{x}(t))(\mathscr{A}_i\mathbf{x}(t) + \mathscr{B}_i\mathbf{u}(t)) \tag{1}$$

with the input $\mathbf{u}(t)$,the state vector $\mathbf{x}(t)$, number of fuzzy rules ω, membership function $\eta_i(\mathbf{x}(t))$ for the i^{th} fuzzy rule, and $\mathscr{A}_i$ and $\mathscr{B}_i$ are the state and input matrices of the i^{th} local linear representation, respectively.

The normalized membership function, denoted by $\eta_i(\mathbf{x}(t))$, is a real-valued function that maps the state of the system, $\mathbf{x}(t)$, to a value in the interval $[0,1]$. It represents the degree of "membership" or "belonging" of $\mathbf{x}(t)$ to the $i-th$ fuzzy set in the Takagi-Sugeno fuzzy model. The membership function can be defined in various ways and the choice of the function depends on the particular application and the characteristics of the system being modeled. Some common forms of membership functions include triangular, trapezoidal, Gaussian, and sigmoid functions. The membership function at $\mathbf{x}(t)$ determines relative importance or weight of the $i-th$ local linear representation in the overall fuzzy model output. If $\eta_i(\mathbf{x}(t))$ is close to 1, then the $i-th$ fuzzy rule is more important or dominant in that region of the state space, and if $\eta_i(\mathbf{x}(t))$ is close to 0, then the $i-th$ fuzzy rule is less important or dominant in that region.

The equivalent representation of Equation (1) would be as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{\omega} \eta_i(\mathbf{z}(t))(\mathscr{A}_i(\mathbf{z}(t))\mathbf{x}(t) + \mathscr{B}_i(\mathbf{z}(t))\mathbf{u}(t)) \tag{2}$$

where the vector of premise variable $\mathbf{z}(t)$ is a $\mathbf{q}-$vector, $\mathbf{z}(t) = \{\mathbf{z}_1(t), \mathbf{z}_2(t), ..., \mathbf{z}_{\mathbf{q}}(t)\}$. It is assumed that the function $\mathbf{z}(t)$ is:

$$\mathbf{z}_k^{min} \leq \mathbf{z}_k \leq \mathbf{z}_k^{max}, \quad \forall k \in \{1, 2, \ldots, \mathbf{q}\}. \tag{3}$$

With the n-vector of state variables $\mathbf{x}(t) = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)^\top$, the function of $\mathbf{z}_k$ can be determined as $\mathbf{z}_k = f(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$. The method for calculating the membership function is as follows:

$$\eta_i(\mathbf{x}(t)) = \prod_{k=1}^{\mathbf{q}} \zeta_k^{\mathfrak{b}}(\mathbf{z}_k) \tag{4}$$

The sector nonlinearity approach is a fuzzy logic method used to represent the nonlinear relationship between input and output variables in fuzzy control systems. It assigns two membership functions, $\mathfrak{b}$ has the value of $\{0,1\}$,ζ^0 and ζ^1, to each premise variable $\mathbf{z}(t)$. ζ^0 representing the lower sector and ζ^1 the upper sector of $\mathbf{z}(t)$. The values of ζ^0 and ζ^1 can be determined.

$$\zeta_k^0(\mathbf{z}_k) = \frac{\mathbf{z}_k^{max} - \mathbf{z}_k}{\mathbf{z}_k^{max} - \mathbf{z}_k^{min}} \tag{5}$$

$$\zeta_k^1(\mathbf{z}_k) = \frac{\mathbf{z}_k - \mathbf{z}_k^{min}}{\mathbf{z}_k^{max} - \mathbf{z}_k^{min}} \tag{6}$$

$$\zeta_k^0 + \zeta_k^1 = 1 \tag{7}$$

with ζ_k^0, $\zeta_k^1 \in [0,1]$.

The convex-sum property is verified by normalized membership functions.

$$\sum_{i=1}^{\omega} \eta_i = 1, \quad \eta_i \geq 0 \tag{8}$$

The $\mathscr{A}$ matrices of the fuzzy controller will be configured as follows:

$$\begin{aligned}
\mathscr{A}_1 &= \mathscr{A}(\mathbf{z}_1^{min}, \mathbf{z}_2^{min}, \mathbf{z}_3^{min}, \ldots, \mathbf{z}_{\mathbf{q}-2}^{min}, \mathbf{z}_{\mathbf{q}-1}^{min}, \mathbf{z}_{\mathbf{q}}^{min}) \\
\mathscr{A}_2 &= \mathscr{A}(\mathbf{z}_1^{min}, \mathbf{z}_2^{min}, \mathbf{z}_3^{min}, \ldots, \mathbf{z}_{\mathbf{q}-2}^{min}, \mathbf{z}_{\mathbf{q}-1}^{min}, \mathbf{z}_{\mathbf{q}}^{max}) \\
\mathscr{A}_3 &= \mathscr{A}(\mathbf{z}_1^{min}, \mathbf{z}_2^{min}, \mathbf{z}_3^{min}, \ldots, \mathbf{z}_{\mathbf{q}-2}^{min}, \mathbf{z}_{\mathbf{q}-1}^{max}, \mathbf{z}_{\mathbf{q}}^{min}) \\
&\vdots \\
\mathscr{A}_{2^{\mathbf{q}-1}} &= \mathscr{A}(\mathbf{z}_1^{min}, \mathbf{z}_2^{max}, \mathbf{z}_3^{max}, \ldots, \mathbf{z}_{\mathbf{q}-2}^{max}, \mathbf{z}_{\mathbf{q}-1}^{max}, \mathbf{z}_{\mathbf{q}}^{max}) \\
\mathscr{A}_{2^{\mathbf{q}-1}+1} &= \mathscr{A}(\mathbf{z}_1^{max}, \mathbf{z}_2^{min}, \mathbf{z}_3^{min}, \ldots, \mathbf{z}_{\mathbf{q}-2}^{min}, \mathbf{z}_{\mathbf{q}-1}^{min}, \mathbf{z}_{\mathbf{q}}^{min}) \\
&\vdots \\
\mathscr{A}_{2^{\mathbf{q}}} &= \mathscr{A}(\mathbf{z}_1^{max}, \mathbf{z}_2^{max}, \mathbf{z}_3^{max}, \ldots, \mathbf{z}_{\mathbf{q}-2}^{max}, \mathbf{z}_{\mathbf{q}-1}^{max}, \mathbf{z}_{\mathbf{q}}^{max})
\end{aligned} \tag{9}$$

3. Reduction Model in applying Takagi-Sugeno Fuzzy Logic

3.1. Fuzzy model reduction

In this section, we emphasize the use of fuzzy logic and reduction models for achieving better results with less complexity. The application of fuzzy logic allows for the handling of uncertainty and imprecise information, while reduction models help to simplify complex data sets.
The choice of matrix $\mathscr{A}$ in a fuzzy model can significantly impact its performance. Using too many matrices $\mathscr{A}$ can result in increased complexity and reduced accuracy. Therefore, it's essential to choose the appropriate matrices $\mathscr{A}$ based on the specific application. One common approach to selecting matrices $\mathscr{A}$ is to use clustering algorithms to group similar data points and generate representative matrices. This method helps reduce the number of matrices $\mathscr{A}$ required while maintaining the accuracy of the fuzzy model. Another approach is to perform a sensitivity analysis on the fuzzy model to determine the effect of different matrices $\mathscr{A}$ on the model's performance. This analysis involves testing the model with different matrices $\mathscr{A}$ and evaluating the resulting performance metrics to determine which matrices $\mathscr{A}$ produce the best outcomes. Ultimately, the choice of matrices $\mathscr{A}$ will depend on the specific requirements of the application and the available data. It's essential to carefully consider the impact of different matrices $\mathscr{A}$ on the fuzzy model's performance to ensure optimal results.

Theorem 1. *The model for the nonlinear system described in equation (1) contains $2^{\mathbf{q}}$ rules, which can reduce the number of rules in a Takagi-Sugeno model with $(\mathbf{q}+1)$ fuzzy rules.*

$$\dot{\mathbf{x}}(t) = \sum_{j=1}^{\mathfrak{N}} \eta_j(\mathbf{x}(t))(\mathscr{A}_j \mathbf{x}(t) + \mathscr{B}_j \mathbf{u}(t)) \tag{10}$$

where $j \in$ the set $\{1,2,3,7,\ldots,2^{\mathbf{q}}-1\}$ that has $(\mathbf{q}+1)$ elements and matrices $\mathscr{A}_i$ in $\mathscr{S} = \{\mathscr{A}_i : i \in \{1,2,3,7,\ldots,2^{\mathbf{q}}-1\}\}$ The $2^{\mathbf{q}}$ fuzzy rules in the TS model of the nonlinear system in equation (1) can be constructed by combinations of fuzzy sets for all input variables. Each of these combinations with the local linear matrices $\mathscr{A}_i$ corresponds to a fuzzy rule in the TS model. However, the TS model can be simplified by using only a subset of these fuzzy rules, which is known as a $(\mathbf{q}+1)$-rules reduction. A membership function η_j assured convex sum property.

Proof: The expressions below can be readily proven due to the property of generating $\mathscr{A}_i$ submatrices in equation (9).

$$\begin{aligned}
\mathscr{A}_2 - \mathscr{A}_1 &= \mathscr{A}_4 - \mathscr{A}_3 = \ldots = \mathscr{A}_{2^{\mathbf{q}}} - \mathscr{A}_{2^{\mathbf{q}}-1} \\
\mathscr{A}_4 - \mathscr{A}_1 &= \mathscr{A}_8 - \mathscr{A}_5 = \ldots = \mathscr{A}_{2^{\mathbf{q}}} - \mathscr{A}_{2^{\mathbf{q}}-3} \\
&\vdots \\
\mathscr{A}_{2^{\mathbf{q}-1}} - \mathscr{A}_1 &= \mathscr{A}_{2^{\mathbf{q}}} - \mathscr{A}_{2^{\mathbf{q}-1}+1}
\end{aligned} \tag{11}$$

The equation that describes the linear relationships in (11) can be expressed as:

$$\mathscr{A}_{2^{\mathbf{m}}} - \mathscr{A}_1 = \mathscr{A}_{2^{\mathbf{m}}\nu} - \mathscr{A}_{2^{\mathbf{m}}\nu-1+1} \tag{12}$$

with $\nu \in \{1,2,3,\ldots,2^{\mathbf{q}-\mathbf{m}}\}$ and $\mathbf{m} \in \{1,2,3,\ldots,\mathbf{q}-1\}$. In the case $\mathbf{m}=1$, the result is:

$$\mathscr{A}_2 - \mathscr{A}_1 = \mathscr{A}_{2^{\mathbf{l}}} - \mathscr{A}_{2^{\mathbf{l}}-1}, \quad \mathbf{l} \in \{1,2,3,\ldots,\mathbf{q}\} \tag{13}$$

Therefore, by replacing the matrices $\mathscr{A}_{2^{\mathbf{m}}}$ with a linear combination of unitary elements, for all values of $\mathbf{m}$ that belong to the set $(2,3,...,\mathbf{q})$, a total of $\mathbf{q}-1$ substitutions can be made. It is a simple task to confirm that this is true for any positive integers $\nu \in \mathbb{N}$ and $\mathbf{m}, \mathbf{l} \in \mathbb{N}^*$.

$$\begin{aligned}
\mathscr{A}_{2^{\mathbf{l}}-1} = \mathscr{A}_{2^{\mathbf{m}}\nu+1} &\Rightarrow 2^{\mathbf{l}} - 1 = 2^{\mathbf{m}}\nu + 1 && (14)\\
&\Rightarrow 2^{\mathbf{l}} = 2^{\mathbf{m}}\nu + 2 && (15)\\
&\Rightarrow 2^{\mathbf{l}-1} - 1 = 2^{\mathbf{m}-1}\nu && (16)\\
&\Rightarrow (\mathbf{l}=1, \nu=0) \quad or \quad (\mathbf{m}=1, \nu = 2^{\mathbf{l}-1}-1) && (17)
\end{aligned}$$

Equation (12) consists of several sets of matrices, where solely the initial group (with $\mathbf{m}=1$) comprise components other than the matrix $\mathscr{A}_1$. Each $\mathbf{m}-th$ set of matrices in equation (12) involves $2^{\mathbf{q}+1-\mathbf{m}}$ matrices and includes all the matrices that appear within the group $(\mathbf{m}+1)-th$ of formulas. Consequently, the quantity of matrices present within $\mathbf{m}-th$ group of formulas that do not have the structure $\mathscr{A}_{2^{\mathbf{l}}}$.

$$2^{\mathbf{q}+1-\mathbf{m}} - [2^{\mathbf{q}-\mathbf{m}} - (\mathbf{q}-\mathbf{m})] - (\mathbf{q}-\mathbf{m}+1) + 1 = 2^{\mathbf{q}-\mathbf{m}} \tag{18}$$

By utilizing both (13) and the overall equation (12) for values of $\mathbf{m}$ ranging from 2 to $\mathbf{q}-1$, each group of equations enables the replacement of $2^{\mathbf{q}-\mathbf{m}}-1$ terms with a sum of linear terms from $\mathbf{m}=\mathbf{q}-1$, via substitution.

$$\begin{aligned}
\mathscr{A}_{2^{\mathbf{q}-1}+1} &= \mathscr{A}_{2^{\mathbf{q}}} - \mathscr{A}_{2^{\mathbf{q}-1}} + \mathscr{A}_1 && (19)\\
&= \mathscr{A}_{2^{\mathbf{q}-1}} + \mathscr{A}_2 - \mathscr{A}_1 - \mathscr{A}_{2^{\mathbf{q}-1}-1} - \mathscr{A}_2 + \mathscr{A}_1 + \mathscr{A}_1 && (20)\\
&= \mathscr{A}_{2^{\mathbf{q}-1}} - \mathscr{A}_{2^{\mathbf{q}-1}-1} + \mathscr{A}_1 && (21)
\end{aligned}$$

If we substitute incrementally for $\mathbf{m}$, starting with 2 and going up to $\mathbf{q}-1$, the total number of substitutions made would be the sum of $\sum_{\mathbf{m}=2}^{\mathbf{q}-1} 2^{\mathbf{q}-\mathbf{m}} - 1 = 2^{\mathbf{q}-1} - \mathbf{q}$. However, none of the matrices that are substituted are part in $\mathscr{S}$, excluding $\mathscr{A}_1$. This is because the other elements of $\mathscr{S}$ are only used in the first set of equations. The $\mathbf{q}+1$ elements of S are not substituted, so when $\mathbf{m}=1$, there are $2^{\mathbf{q}-1} - 1 - (\mathbf{q}+1) = 2^{\mathbf{q}-1} - \mathbf{q}$ substitutions. Thus, the total number of substitutions made would be:

$$\mathbf{q} - 1 + 2^{\mathbf{q}-1} - \mathbf{q} + 2^{\mathbf{q}-1} - \mathbf{q} = 2^{\mathbf{q}} - \mathbf{q} - 1 \tag{22}$$

By including the $\mathbf{q}+1$ elements of set $\mathscr{S}$, we can calculate the number of rules, denoted as r, in the TS fuzzy model by taking 2 to the power of $\mathbf{q}$. We can replace each matrix $\mathscr{A}_i$ that is not in set $\mathscr{S}$, in equation (2), with a unitary linear combination. This will enable us to prove the theorem where η_j satisfies the convex sum property. This completes the proof.

3.2. Fuzzy controller design

The control signal is given by:

$$\mathbf{u}(t) = -\mathscr{F}_j\mathbf{x}(t) \qquad j = \{1,2,3,7,\ldots,2^{\mathbf{q}}-1\}. \quad (23)$$

Theorem 2. *The PDC (Parallel Distributed Compensation) controller (23) ensures the asymptotic stability of the system (1) if the LMI constraints presented below are fulfilled with a shared positive definite matrix H and matrices $\mathscr{G}_{\mathbf{n}}$.*

$$\begin{cases} \Phi_{\mathbf{mm}} < 0, & \forall \mathbf{m},\mathbf{n} \in \{1,2,3,7,\ldots,2^{\mathbf{q}}-1\} \\ \Phi_{\mathbf{mn}} + \Phi_{\mathbf{nm}} < 0, & \mathbf{m} < \mathbf{n};\ \mathbf{m},\mathbf{n} \in \{1,2,3,7,\ldots,2^{\mathbf{q}}-1\} \end{cases} \quad (24)$$

with $\Phi_{\mathbf{mn}} = \mathscr{A}_{\mathbf{m}}\mathscr{X} - \mathscr{B}_{\mathbf{m}}\mathscr{G}_{\mathbf{n}} + \mathscr{X}\mathscr{A}_{\mathbf{m}}^\top - \mathscr{G}_{\mathbf{n}}^\top\mathscr{B}_{\mathbf{m}}^\top$. As a result, the control gains of the PDC controller can be deduced in the following manner:

$$\mathscr{F}_{\mathbf{n}} = \mathscr{G}_{\mathbf{n}}\mathscr{X}^{-1} \quad (25)$$

Proof: Let us contemplate a Lyapunov function candidate having a positive definite matrix $\mathscr{P}$:

$$V(\mathbf{x}) = \mathbf{x}^\top(t)\mathscr{P}\mathbf{x}(t). \quad (26)$$

The above function can be obtained by following the derivation process below:

$$\dot{V}(\mathbf{x}) = \mathbf{x}^\top(t)\left(\dot{\mathscr{P}}\mathbf{x}(t) + \mathscr{P}\dot{\mathbf{x}}(t)\right) + \dot{\mathbf{x}}^\top(t)\mathscr{P}\mathbf{x}(t). \quad (27)$$

It can be deduced that

$$\begin{aligned} \dot{V}(\mathbf{x}) = {} & \sum_{\mathbf{m}=1}^{2^{\mathbf{q}}-1}\sum_{\mathbf{n}=1}^{2^{\mathbf{q}}-1} \eta_{\mathbf{m}}(\mathbf{z}(t))\eta_{\mathbf{n}}(\mathbf{z}(t))\mathbf{x}^\top(t) \\ & \left[(\mathscr{A}_{\mathbf{m}} - \mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{n}})^\top\mathscr{P} + \mathscr{P}(\mathscr{A}_{\mathbf{m}} - \mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{n}})\right]\mathbf{x}(t) \\ = {} & \sum_{\mathbf{m}=1}^{2^{\mathbf{q}}-1} \eta_{\mathbf{m}}^2(\mathbf{z}(t))\mathbf{x}^\top(t) \\ & \left[(\mathscr{A}_{\mathbf{m}} - \mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{m}})^\top\mathscr{P} + \mathscr{P}(\mathscr{A}_{\mathbf{m}} - \mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{m}})\right]\mathbf{x}(t) \\ & + 2\sum_{\mathbf{m}=1}^{2^{\mathbf{q}}-1}\sum_{\mathbf{m}<\mathbf{n}}^{2^{\mathbf{p}}} \eta_{\mathbf{m}}(\mathbf{z}(t))\eta_{\mathbf{n}}(\mathbf{z}(t))\mathbf{x}^\top(t) \\ & \times\left[\left(\frac{(\mathscr{A}_{\mathbf{m}} - \mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{n}}) + (\mathscr{A}_{\mathbf{n}} - \mathscr{B}_{\mathbf{n}}\mathscr{F}_{\mathbf{m}})}{2}\right)^\top\mathscr{P}\right]\mathbf{x}(t) \\ & + 2\sum_{\mathbf{m}=1}^{2^{\mathbf{q}}-1}\sum_{\mathbf{m}<\mathbf{n}}^{2^{\mathbf{p}}} \eta_{\mathbf{m}}(\mathbf{z}(t))\eta_{\mathbf{n}}(\mathbf{z}(t))\mathbf{x}^\top(t) \\ & \times\left[\mathscr{P}\frac{(\mathscr{A}_{\mathbf{m}} - \mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{n}}) + (\mathscr{A}_{\mathbf{n}} - \mathscr{B}_{\mathbf{n}}\mathscr{F}_{\mathbf{m}})}{2}\right]\mathbf{x}(t) \end{aligned} \quad (28)$$

We have:

$$\begin{cases} \mathscr{A}_{\mathbf{m}}^\top\mathscr{P} + \mathscr{P}\mathscr{A}_{\mathbf{m}} - \mathscr{F}_{\mathbf{m}}^\top\mathscr{B}_{\mathbf{m}}^\top\mathscr{P} - \mathscr{P}\mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{m}} < 0 \\ \mathscr{A}_{\mathbf{m}}^\top\mathscr{P} + \mathscr{P}\mathscr{A}_{\mathbf{m}} - \mathscr{F}_{\mathbf{n}}^\top\mathscr{B}_{\mathbf{m}}^\top\mathscr{P} - \mathscr{P}\mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{n}} + \mathscr{A}_{\mathbf{n}}^\top\mathscr{P} + \mathscr{P}\mathscr{A}_{\mathbf{n}} \\ -\mathscr{F}_{\mathbf{m}}^\top\mathscr{B}_{\mathbf{n}}^\top\mathscr{P} - \mathscr{P}\mathscr{B}_{\mathbf{n}}\mathscr{F}_{\mathbf{m}} < 0 \end{cases} \quad (29)$$

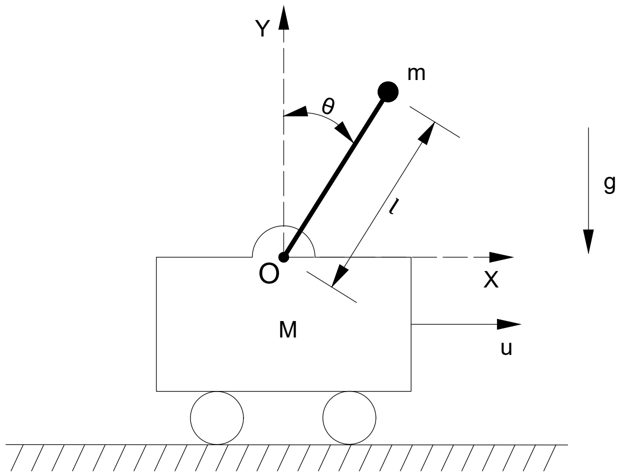

Figure 1. Modeling of an inverted pendulum.

If we multiply $\mathscr{X} = \mathscr{P}^{-1}$ on both the left and right-hand side of equation (29), we obtain:

$$\begin{cases} \mathscr{X}\mathscr{A}_{\mathbf{m}}^\top + \mathscr{A}_{\mathbf{m}}\mathscr{X} - \mathscr{X}\mathscr{F}_{\mathbf{m}}^\top\mathscr{B}_{\mathbf{m}}^\top - \mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{m}}\mathscr{X} < 0 \\ \mathscr{X}\mathscr{A}_{\mathbf{m}}^\top + \mathscr{A}_{\mathbf{m}}\mathscr{X} - \mathscr{X}\mathscr{F}_n^\top\mathscr{B}_{\mathbf{m}}^\top - \mathscr{B}_{\mathbf{m}}\mathscr{F}_{\mathbf{n}}\mathscr{X} + \mathscr{X}\mathscr{A}_{\mathbf{n}}^\top \\ +\mathscr{A}_{\mathbf{n}}\mathscr{X} - \mathscr{X}\mathscr{F}_{\mathbf{m}}^\top\mathscr{B}_n^\top - \mathscr{B}_{\mathbf{n}}\mathscr{F}_{\mathbf{m}}\mathscr{X} < 0 \end{cases} \quad (30)$$

For $\mathscr{X} > 0$ and let $\mathscr{G}_n = \mathscr{F}_n\mathscr{X}$. Replacing the expression obtained into the previous inequality leads to:

$$\begin{cases} \mathscr{X}\mathscr{A}_{\mathbf{m}}^\top + \mathscr{A}_{\mathbf{m}}\mathscr{X} - \mathscr{G}_{\mathbf{m}}^\top\mathscr{B}_{\mathbf{m}}^\top - \mathscr{B}_{\mathbf{m}}\mathscr{G}_{\mathbf{m}} < 0 \\ \mathscr{X}\mathscr{A}_{\mathbf{m}}^\top + \mathscr{A}_{\mathbf{m}}\mathscr{X} - \mathscr{G}_{\mathbf{n}}^\top\mathscr{B}_{\mathbf{m}}^\top - \mathscr{B}_{\mathbf{m}}\mathscr{G}_{\mathbf{n}} + \mathscr{X}\mathscr{A}_{\mathbf{n}}^\top + \mathscr{A}_{\mathbf{n}}\mathscr{X} \\ -\mathscr{G}_{\mathbf{m}}^\top\mathscr{B}_{\mathbf{n}}^\top - \mathscr{B}_{\mathbf{n}}\mathscr{G}_{\mathbf{m}} < 0 \end{cases} \quad (31)$$

with $\Phi_{\mathbf{mn}} = \mathscr{A}_{\mathbf{m}}\mathscr{X} - \mathscr{B}_{\mathbf{m}}\mathscr{G}_{\mathbf{n}} + \mathscr{X}\mathscr{A}_{\mathbf{m}}^\top - \mathscr{G}_{\mathbf{n}}^\top\mathscr{B}_{\mathbf{m}}^\top$, the LMI conditions can be derived by using the result of the previous analysis.

4. Control System Modeling and Design using Fuzzy Logic: An Inverted Pendulum on a Cart Example

4.1. Inverted Pendulum System Modeling and Analysis

The research employs a simple inverted pendulum model as depicted in Figure 1. The model comprises a pendulum of mass m (kg) and a cart of mass M (kg). The connecting rod has a length l (m) and the rotational angle of the pendulum from the y-axis is denoted by θ (rad). The force exerted on the cart in the x-axis is $\mathbf{u}$, while g is the gravitational acceleration vector. We have:

$$\begin{cases} \bar{x} & = x_M + l sin\theta \\ \bar{y} & = l cos\theta \end{cases} \quad (32)$$

with x_M is the displacement of the cart.
As the mass of the connecting rod is negligible compared to the cart, the total energy of the system can be expressed as follows:

$$L = \frac{1}{2}(M+m)\dot{x}_M^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{x}_M cos\theta - mglcos\theta. \quad (33)$$

The Euler-Lagrange equation is used to derive the kinematic equation of the inverted pendulum:

$$\frac{d}{dt}(\frac{\delta L}{\delta \dot{q_m}}) - \frac{\delta L}{\delta q_m} = Q_m. \quad (34)$$

From equation (34), we have:

$$\begin{cases} (M+m)\ddot{x}_M + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = \mathbf{u} \\ l\ddot{\theta} + \ddot{x}_M cos\theta - gsin\theta = 0 \end{cases} \quad (35)$$

We can obtain the equation that describes the angular dynamics of the pendulum by manipulating the set of equations (35), which can be expressed as follows:

$$\ddot{\theta} = \frac{(M+m)gsin\theta - ml\dot{\theta}^2 sin\theta cos\theta - \mathbf{u}cos\theta}{l[m(1-cos^2\theta)+M]}. \quad (36)$$

Define $\mathbf{x} = \begin{bmatrix}\theta & \dot{\theta}\end{bmatrix}^T$, then the expression of equation (36) can be written in the following form:

$$\dot{\mathbf{x}} = \mathscr{A}\mathbf{x} + \mathscr{B}\mathbf{u} \quad (37)$$

where

$$\mathscr{A} = \begin{bmatrix} 0 & 1 \\ \frac{(M+m)gsin\theta}{\theta l[m(1-cos^2\theta)+M]} & \frac{-ml\dot{\theta}sin\theta cos\theta}{l[m(1-cos^2\theta)+M]} \end{bmatrix},$$
$$\mathscr{B} = \begin{bmatrix} 0 \\ \frac{-cos\theta}{l[m(1-cos^2\theta)+M} \end{bmatrix}.$$

4.2. TS controller design

The initial step is to define the premise variables as given below:

$$\begin{aligned} \mathbf{z}_1 &= \frac{1}{L[m(1-cos^2\theta)+M}, \\ \mathbf{z}_2 &= \frac{sin\theta}{\theta}, \\ \mathbf{z}_3 &= cos\theta, \\ \mathbf{z}_4 &= \dot{\theta}sin\theta. \end{aligned} \quad (38)$$

Substituting the premise variables $\mathbf{z}$ into the matrices $\mathscr{A}$, $\mathscr{B}$ above we obtain:

$$\mathscr{A} = \begin{bmatrix} 0 & 1 \\ (M+m)g\mathbf{z}_1\mathbf{z}_2 & -ml\mathbf{z}_1\mathbf{z}_3\mathbf{z}_4 \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} 0 \\ -\mathbf{z}_1\mathbf{z}_3 \end{bmatrix}.$$

Define:

$$\mathbf{h}_{i0} = \frac{\mathbf{z}_{imax} - \mathbf{z}_i}{\mathbf{z}_{imax} - \mathbf{z}_{imin}}, \quad \mathbf{h}_{i1} = 1 - \mathbf{h}_{i0} \quad (i = 1,2,3,4) \quad (39)$$

The membership functions of the TS fuzzy model can be defined as follows:

$$\begin{aligned} \eta_1 &= \mathbf{h}_{10} * \mathbf{h}_{20} * \mathbf{h}_{30} * \mathbf{h}_{40}, & \eta_9 &= \mathbf{h}_{10} * \mathbf{h}_{20} * \mathbf{h}_{30} * \mathbf{h}_{41}, \\ \eta_2 &= \mathbf{h}_{11} * \mathbf{h}_{20} * \mathbf{h}_{30} * \mathbf{h}_{40}, & \eta_{10} &= \mathbf{h}_{11} * \mathbf{h}_{20} * \mathbf{h}_{30} * \mathbf{h}_{41}, \\ \eta_3 &= \mathbf{h}_{10} * \mathbf{h}_{21} * \mathbf{h}_{30} * \mathbf{h}_{40}, & \eta_{11} &= \mathbf{h}_{10} * \mathbf{h}_{21} * \mathbf{h}_{30} * \mathbf{h}_{41}, \\ \eta_4 &= \mathbf{h}_{11} * \mathbf{h}_{21} * \mathbf{h}_{30} * \mathbf{h}_{40}, & \eta_{12} &= \mathbf{h}_{11} * \mathbf{h}_{21} * \mathbf{h}_{30} * \mathbf{h}_{41}, \end{aligned} \quad (40)$$

$$\begin{aligned} \eta_5 &= \mathbf{h}_{10} * \mathbf{h}_{20} * \mathbf{h}_{31} * \mathbf{h}_{40}, & \eta_{13} &= \mathbf{h}_{10} * \mathbf{h}_{20} * \mathbf{h}_{31} * \mathbf{h}_{41}, \\ \eta_6 &= \mathbf{h}_{11} * \mathbf{h}_{20} * \mathbf{h}_{31} * \mathbf{h}_{40}, & \eta_{14} &= \mathbf{h}_{11} * \mathbf{h}_{20} * \mathbf{h}_{31} * \mathbf{h}_{41}, \\ \eta_7 &= \mathbf{h}_{10} * \mathbf{h}_{21} * \mathbf{h}_{31} * \mathbf{h}_{40}, & \eta_{15} &= \mathbf{h}_{10} * \mathbf{h}_{21} * \mathbf{h}_{31} * \mathbf{h}_{41}, \\ \eta_8 &= \mathbf{h}_{11} * \mathbf{h}_{21} * \mathbf{h}_{31} * \mathbf{h}_{40}, & \eta_{16} &= \mathbf{h}_{11} * \mathbf{h}_{21} * \mathbf{h}_{31} * \mathbf{h}_{41}. \end{aligned}$$

The model originally comprised of 16 fuzzy rules, however, Theorem 1 enables reducing the rules to only five, namely the 1st, 2nd, 3rd, 7th, and 15th rules.

$$\begin{aligned} \eta_1 &= \mathbf{h}_{10} * \mathbf{h}_{20} * \mathbf{h}_{30} * \mathbf{h}_{40}, \\ \eta_2 &= \mathbf{h}_{11} * \mathbf{h}_{20} * \mathbf{h}_{30} * \mathbf{h}_{40}, \\ \eta_3 &= \mathbf{h}_{10} * \mathbf{h}_{21} * \mathbf{h}_{30} * \mathbf{h}_{40}, \\ \eta_7 &= \mathbf{h}_{10} * \mathbf{h}_{21} * \mathbf{h}_{31} * \mathbf{h}_{40}, \\ \eta_{15} &= \mathbf{h}_{10} * \mathbf{h}_{21} * \mathbf{h}_{31} * \mathbf{h}_{41}. \end{aligned} \quad (41)$$

The control signal of TS 16 rules model:

$$\begin{aligned} \mathbf{u}_{16r}(t) = &-(\mathscr{F}_1\eta_1 + \mathscr{F}_2\eta_2 + \mathscr{F}_3\eta_3 + \mathscr{F}_4\eta_4 + \mathscr{F}_5\eta_5 \\ &+ \mathscr{F}_6\eta_6 + \mathscr{F}_7\eta_7 + \mathscr{F}_8\eta_8 + \mathscr{F}_9\eta_9 + \mathscr{F}_{10}\eta_{10} \\ &+ \mathscr{F}_{11}\eta_{11} + \mathscr{F}_{12}\eta_{12} + \mathscr{F}_{13}\eta_{13} + \mathscr{F}_{14}\eta_{14} + \mathscr{F}_{15}\eta_{15} \\ &+ \mathscr{F}_{16}\eta_{16})\mathbf{x}(t). \end{aligned} \quad (42)$$

The control signal of TS reduction rules model:

$$\mathbf{u}_{5r}(t) = -(\mathscr{F}_1\eta_1 + \mathscr{F}_2\eta_2 + \mathscr{F}_3\eta_3 + \mathscr{F}_7\eta_7 + \mathscr{F}_{15}\eta_{15})\mathbf{x}(t). \quad (43)$$

5. Result

The system parameters for the pendulum in this study were chosen as follows: $m = 0.2$ kg, $g = 9.8\ m/s^2$, $M = 1$ kg, $l = 1$ m, and $\theta_0 \in [-\pi/2; \pi/2]$. Using these parameters, the values of $\mathbf{z}_{imax}$ and $\mathbf{z}_{imin}$ were determined, and subsequently, the values of $\mathbf{h}_{i0}$ and $\mathbf{h}_{i1}$ (39) and the membership functions (40) were calculated. The feedback gains $\mathscr{F}_\mathbf{m}$ were also computed by solving the LMI using MATLABs Robust Control Toolbox. The control signal $\mathbf{u}$ was determined (42,43). The simulation results were then computed for different initial angles θ_0 within the specified limits.

In these figures, the blue and red lines are the angle and its velocity in two cases: the first case uses fuzzy controller with reduction model (5 rules) and the other case uses fuzzy controller with 16 rules. Fig 2, 3 show angle and angular velocity of pendulum with $\theta_0 = 15°$ and fig 4, 5 show in the case with $\theta_0 = -15°$.

The results in fig 2 suggest that both the reduction and full TS fuzzy models effectively stabilize the inverted pendulum, as evidenced by the convergence of the deflection angle after about 2 seconds. However, the use of the full 16 fuzzy rules results in a smoother graph of the deviation angle and angular speed than the reduced rule version. A subtle bend is observed in the blue line at 1.25 seconds, which is also reflected

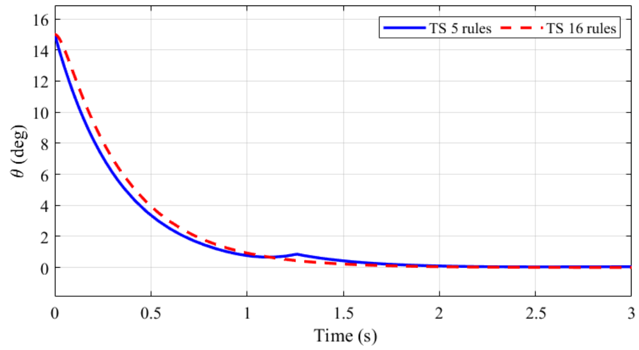

Figure 2. Angle of pendulum with $\theta_0 = 15°$.

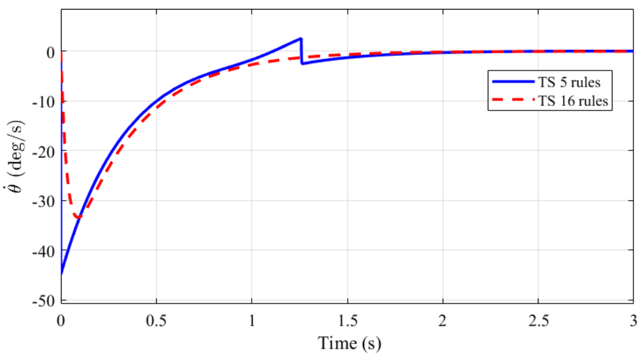

Figure 3. Angular velocity of pendulum with $\theta_0 = 15°$.

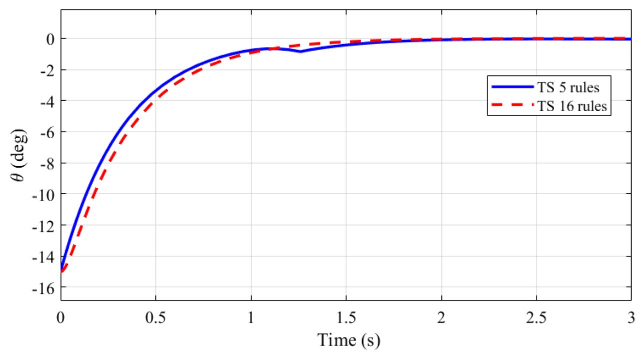

Figure 4. Angle of pendulum with $\theta_0 = -15°$.

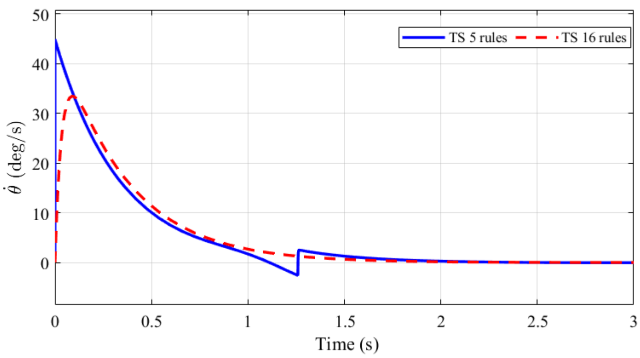

Figure 5. Angular velocity of pendulum with $\theta_0 = -15°$.

in the angular speed in fig 3. Similarly, for the initial angle of -15 degrees in fig 4, both models demonstrate quick convergence, and the difference between the two curves with 16 and 5 fuzzy rules is negligible. The results indicate that the proposed method is effective in ensuring the stability of the pendulum, as both cases yield equally good outcomes despite the reduction in model complexity.

6. Conclusion

In conclusion, Takagi-Sugeno (TS) fuzzy control methodology is a powerful approach for controlling nonlinear systems, particularly those that are difficult to model using traditional linear control methods. However, reducing the number of fuzzy rules in TS control is necessary to simplify control system design, reduce computational burden, and improve control performance and stability. This paper proposed a novel method to format the TS fuzzy model while reducing the number of fuzzy rules, which was applied to the example of an inverted pendulum model. The simulation results demonstrated the efficacy of the proposed approach in stabilizing the system and providing a more straightforward approach to designing a controller.
The reduction in the number of fuzzy rules also has implications for the extension of the control system to new applications, making it more flexible and adaptable to changing requirements. Future research could explore the application of the proposed approach to other nonlinear systems and investigate the effects of different membership functions on the performance of TS fuzzy control. In summary, the proposed approach offers a promising method for improving the effectiveness and efficiency of TS fuzzy control in controlling complex nonlinear systems.

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